Topics in Logic and Complexity

Handout 5

Anuj Dawar

http://www.cl.cam.ac.uk/teaching/2223/L15

Constraint Satisfaction Problems

Example:
Can we find x, y, z such that

$$\begin{aligned} x+y+z &\geq 4 \\ x-y &= 3 \\ z &\leq 2 \\ x &= 1 \end{aligned}$$

Constraint Satisfaction Problems

In general a *constraint satisfaction problem (CSP)* is specified by:

- A collection V of *variables*.
- For each variable $x \in V$ a *domain* D_v of possible *values*.
- A collection of *constraints* each of which consists of a tuple $(x_1, \ldots, x_r)$ of variables and a set

$$S \subseteq D_{x_1} \times \cdots \times D_{x_r}$$

of permitted combinations of values.

We consider *finite-domain* CSP, where the sets D_x are *finite*.
We further make the simplifying assumption that there is a *single domain* D, with $D_x = D$ for all $x \in V$.

Constraint Satisfaction Problems

In general a *constraint satisfaction problem (CSP)* is specified by:

- A collection V of *variables*.
- A domain D of *values*
- A collection of *constraints* each of which consists of a tuple $(x_1, \ldots, x_r)$ of variables and a set $S \subseteq D^r$ of permitted combinations of values.

The problem is to *decide* if there is an assignment

$$\eta : V \to D$$

such that for each constraint $C = (\mathrm{x}, S)$ we have

$$\eta(\mathrm{x}) \in S.$$

Example - Boolean Satisfiability

Consider a Boolean formula ϕ in *conjunctive normal form* (CNF).
This can be seen as *CSP* with

- V the set of variables occurring in ϕ
- $D = \{0, 1\}$
- a *constraint* for each *clause* of ϕ.

The clause $x \vee y \vee \bar{z}$ gives the constraint $(x, y, z), S$ where

$$S = \{(0,0,0), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

Structure Homomorphism

Fix a relational signature σ (no function or constant symbols).
Let $\mathbb{A}$ and $\mathbb{B}$ be two σ-structures.
A *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a function $h : A \to B$ such that for each relation $R \in \sigma$ and each tuple a

$$\mathrm{a} \in R^{\mathbb{A}} \quad \Rightarrow \quad h(\mathrm{a}) \in R^{\mathbb{B}}$$

The problem of deciding, given $\mathbb{A}$ and $\mathbb{B}$ whether there is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ is NP-complete. Why?

Homomorphism and CSP

Given a CSP with variables V, domain D and constraints $\mathcal{C}$, let σ be a signature with a relation symbol R_S of arity r for each distinct relation $S \subseteq D^r$ occurring in $\mathcal{C}$.

Let $\mathbb{B}$ be the σ-structure with universe D where each R_S is interpreted by the relation S

Let $\mathbb{A}$ be the structure with universe V where R_S is interpreted as the set of all tuples x for which $(\mathrm{x}, S) \in \mathcal{C}$.

Then, the CSP is solvable *if, and only if,* there is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

Complexity of CSP

Write $\mathbb{A} \longrightarrow \mathbb{B}$ to denote that *there is* a homomorphism from $\mathbb{A}$ to $\mathbb{B}$.

The problem of determining, given $\mathbb{A}$ and $\mathbb{B}$, whether $\mathbb{A} \longrightarrow \mathbb{B}$ is *NP-complete*, and can be decided in time $O(|B|^{|A|})$.

So, for a fixed structure $\mathbb{A}$, the problem of deciding membership in the set

$$\{\mathbb{B} \mid \mathbb{A} \longrightarrow \mathbb{B}\}$$

is in P.

Non-uniform CSP

On the other hand, for a fixed structure $\mathbb{B}$, we define the *non-uniform CSP* with template $\mathbb{B}$, written $\mathsf{CSP}(\mathbb{B})$ as the class of structures

$$\{\mathbb{A} \mid \mathbb{A} \longrightarrow \mathbb{B}\}$$

The complexity of $\mathsf{CSP}(\mathbb{B})$ depends on the particular structure $\mathbb{B}$.

The problem is always in NP. For some $\mathbb{B}$, it is in P and for others it is NP-complete

Example - 3-SAT

Let $\mathbb{B}$ be a structure with universe $\{0,1\}$ and *eight* relations

$$R_{000}, R_{001}, R_{010}, R_{011}, R_{100}, R_{101}, R_{110}, R_{111}$$

where R_{ijk} is defined to be the relation

$$\{0,1\}^3 \setminus \{(i,j,k)\}.$$

Then, CSP($\mathbb{B}$) is *essentially* the problem of determining satisfiability of Boolean formulas in *3-CNF*.

Example - 3-Colourability

Let K_n be the *complete* simple undirected graph on n vertices.

Then, an undirected simple graph is in $\mathrm{CSP}(K_3)$ *if, and only if,* it is *3-colourable*.

$\mathrm{CSP}(K_3)$ is NP-complete.

On the other hand, $\mathrm{CSP}(K_2)$ is in P.

Example - 3XOR-SAT

Let $\mathbb{B}$ be a structure with universe $\{0, 1\}$ and *two* ternary relations

$$R_0 \text{ and } R_1.$$

where R_i is the collection of triples $(x, y, z) \in \{0, 1\}^3$ such that

$$x + y + z \equiv i \pmod 2$$

Then, CSP($\mathbb{B}$) is *essentially* the problem of determining satisfiability of Boolean formulas in *3-XOR-CNF*.
This problem is in P.

Schaefer's theorem

Schaefer (1978) proved that if $\mathbb{B}$ is a structure on domain $\{0, 1\}$, then CSP($\mathbb{B}$) is in P if one of the following cases holds:

1. Each relation of $\mathbb{B}$ is 0-*valid*.
2. Each relation of $\mathbb{B}$ is 1-*valid*.
3. Each relation of $\mathbb{B}$ is *bijunctive*.
4. Each relation of $\mathbb{B}$ is *Horn*.
5. Each relation of $\mathbb{B}$ is *dual Horn*.
6. Each relation of $\mathbb{B}$ is *affine*.

In all other cases, CSP($\mathbb{B}$) is *NP-complete*.

Hell-Nešetřil theorem

Let H be a *simple, undirected graph*.

Hell and Nešetřil (1990) proved that CSP(H) is in P if one of the following holds

1. H is *edgeless*
2. H is *bipartite*

In all other cases, CSP(H) is *NP-complete*.

Feder-Vardi conjecture

Feder and Vardi (1993) conjectured that for *every* finite relational structure $\mathbb{B}$:

either $\mathrm{CSP}(\mathbb{B})$ *is in* P *or it is* NP*-complete.*

Ladner (1975) showed that for any *languages* L and K, if $L \leq_P K$ and $K \not\leq_P L$, then there is a language M with

$$L \leq_P M \leq_P K \text{ and } K \not\leq_P M \text{ and } M \not\leq_P L$$

Corollary: if $\mathrm{P} \neq \mathrm{NP}$ then there are problems in NP that are neither in P nor NP-complete.

Bulatov-Zhuk theorem

Bulatov and Zhuk (2017) independently proved the Feder-Vardi *dichotomy conjecture*.

The result came after a twenty-year development of the theory of CSP based on *universal algebra*.

The complexity of CSP($\mathbb{B}$) can be completely classified based on the identitites satisfied by the *algebra of polymorphisms* of the structure $\mathbb{B}$.

Polymorphisms

For a pair of structures $\mathbb{A}$ and $\mathbb{B}$ over the same relational structure σ, we write $\mathbb{A} \times \mathbb{B}$ for their *Cartesian product*.
This is defined to be the σ-structure with universe $A \times B$ so that for any r-ary $R \in \sigma$:

$$((a_1, b_1), \dots, (a_r, b_r)) \in R^{\mathbb{A} \times \mathbb{B}} \quad \text{if, and only if,}$$

$$(a_1, \dots, a_r) \in R^{\mathbb{A}} \textit{ and } (b_1, \dots, b_r) \in R^{\mathbb{B}}.$$

Note: we always have $\mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{A}$ and $\mathbb{A} \times \mathbb{B} \longrightarrow \mathbb{B}$

Polymorphisms

We define the *kth power* of $\mathbb{B}$, written $\mathbb{B}^k$ to be the Cartesian product of $\mathbb{B}$ to itself.

For a structure $\mathbb{B}$, a *k-ary polymorphism* of $\mathbb{B}$ is a homomorphism

$$h : \mathbb{B}^k \longrightarrow \mathbb{B}$$

The collection of all polymorphisms of $\mathbb{B}$ forms an *algebraic structure* called the *clone of polymorphisms* of $\mathbb{B}$.

Algebraic properties of this clone *determine* the *complexity* of CSP($\mathbb{B}$).

CSP and MSO

For any fixed finite structure $\mathbb{B}$, the class of structures $\mathrm{CSP}(\mathbb{B})$ is definable in *existential MSO*.

Let $b_1, \ldots, b_n$ *enumerate* the elements of $\mathbb{B}$.

$$\begin{aligned}\exists X_1 \cdots \exists X_n \quad & \forall x \bigvee_i X_i(x) \wedge \\ & \forall x \bigwedge_{i \neq j} X_i(x) \rightarrow \neg X_j(x) \wedge \\ & \bigwedge_{R \in \sigma} \forall x_1 \cdots \forall x_r \big(R(x_1 \cdots x_r) \rightarrow \bigvee_{(b_{i_1} \cdots b_{i_r}) \in R^{\mathbb{B}}} \bigwedge_j X_{i_j}(x_j) \big)\end{aligned}$$

A structure $\mathbb{A}$ satisfies this sentence *if, and only if,* $\mathbb{A} \longrightarrow \mathbb{B}$.

k-local Consistency Algorithm

For a positive integer k we define an algorithm called the *k-consistency algorithm* for testing whether $\mathbb{A} \longrightarrow \mathbb{B}$.

Let S_0 be the collection of all *partial homomorphisms* $h : \mathbb{A} \hookrightarrow \mathbb{B}$ with *domain size* k.

Given a set $S \subseteq S_0$, say that $h \in S$ is *extendable* in S if

> *for each restriction g of h to $k-1$ elements and eacch $a \in A$, there is an $h' \in S$ that extends g and whose domain includes a.*

k-local Consistency Algorithm

The k-consistency algorithm can now be described as follows

1. $S := S_0$;
2. $S' := \{h \in S \mid h \text{ is extendable in } S\}$
3. if $S' = \emptyset$ then reject
4. else if $S' = S$ then accept
5. else goto 2.

If this algorithm rejects then $\mathbb{A} \not\longrightarrow \mathbb{B}$.
If the algorithm accepts, we can't be sure.

Bounded Width CSP

We say that $CSP(\mathbb{B})$ has *width k* if the k-consistency algorithm *correctly* determines for each $\mathbb{A}$ whether or not $\mathbb{A} \longrightarrow \mathbb{B}$.

We say that $CSP(\mathbb{B})$ has *bounded width* if there is some k such that it has width k.

Note: If $CSP(\mathbb{B})$ has bounded width, it is solvable in *polynomial time*.

$\mathrm{CSP}(K_2)$ has width 3.
$\mathrm{CSP}(K_3)$ has *unbounded* width.

Definability in LFP

If CSP($\mathbb{B}$) is of bounded width, there is a sentence of LFP that *defines* it.

The k-consistency algorithm is computing the *largest* set $S \subseteq S_0$ such that every $h \in S$ is extendable in S.
This can be defined as the *greatest fixed point* of an operator definable in *first-order logic*.

Exercise: prove it!

Fact: If CSP($\mathbb{B}$) is definable in LFP then it has *bounded width*.

Fact: There are $\mathbb{B}$ for which CSP($\mathbb{B}$) is in P, but not of bounded width.

Near-Unanimity Polymorphisms

For $k \geq 3$, a function $f : B^k \to B$ is said to be a *near-unanimity* (NU) function if for all $a, b \in B$

$$f(a, \dots, a, b) = f(a, \dots, b, a) = \cdots = f(b, \dots, a, a) = a.$$

Say $\mathbb{B}$ has a *near-unanimity polymorphism* of arity k if there is a k-ary near-unanimity function that is a *polymorphism* of $\mathbb{B}$.

Fact: if $\mathbb{B}$ has a NU polymorphism of arity k then for every $l > k$, it has a NU polymorphism of arity l.

If $g : \mathbb{B}^k \to \mathbb{B}$ is a NU polymorphism, define

$$h(x_1, \dots, x_l) = g(x_1, \dots, x_k)$$

Near-Unanimity and Bounded Width

Theorem
If $\mathbb{B}$ has a NU polymorphism of arity k, then $\mathrm{CSP}(\mathbb{B})$ has *width* k.

Suppose S is a *non-empty* set of partial homomorphisms $h : \mathbb{A} \hookrightarrow \mathbb{B}$, each of which is *extendable* in S.

We can use this *and* the NU polymorphisms of $\mathbb{B}$ to construct a *total* homomorphism $g : \mathbb{A} \to \mathbb{B}$.