1. In the lecture we saw an illustration of a construction to show that acyclicity of graphs is not definable in first-order logic. Write out a proof of this result.

Prove that acyclicity is not definable in $\text{Mon.}\Sigma^1_1$. Is it definable in $\text{Mon.}\Pi^1_1$?

2. Prove (using Hanf’s theorem or otherwise) that 3-colourbility of graphs is not definable in first-order logic.

Graph 3-colourability (and, indeed, 2-colourability) are definable in $\text{Mon.}\Sigma^1_1$. Can you show they are not definable in $\text{Mon.}\Pi^1_1$? Are they definable in universal second-order logic?

3. Prove the lemma stated in the lecture that any formula that is positive in the relation symbol $R$ defines a monotone operator.

4. Prove that the formula of LFP $\neg[lfp_{R,x}\neg\phi(R/\neg R)](x)$, where $\phi(R/\neg R)$ denotes the result of replacing all occurrences of $R$ in $\phi$ by $\neg R$, defines the greatest fixed point of the operator defined by $\phi$.

5. In the lectures, we saw how definitions by simultaneous induction can be replaced by a single application of the $lfp$ operator. In this exercise, you are asked to show the same for nested applications of the $lfp$ operator.

Suppose $\phi(x, y, S, T)$ is a formula in which the relational variables $S$ (of arity $s$) and $T$ (of arity $t$) only appear positively, and $x$ and $y$ are tuples of variables of length $s$ and $t$ respectively. Show that (for any $t$-tuple of terms $t$) the predicate expression

$$[lfp_{S,x}(lfp_{T,y}\phi(t)))]$$

is equivalent to an expression with just one application of $lfp$.

6. Consider a vocabulary consisting of two unary relations $P$ and $O$, one binary relation $E$ and two constants $s$ and $t$. We say that a structure $A = (A, P, O, E, s, t)$ in this vocabulary is an arena if $P \cup O = A$ and $P \cap O = \emptyset$. That is, $P$ and $O$ partition the universe into two disjoint sets.

An arena defines the following game played between a player and an opponent. The game involves a token that is initially placed on the element $s$. At each move, if the token is currently on an element of $P$ it is player who plays and if it is on an element of $O$, it is opponent who plays. At each move, if the token is on an element $a$, the one who plays choses an element $b$ such that $(a, b) \in E$ and moves the token from $a$ to $b$. If the token reaches $t$ at any point then player has won the game.
We define \textsc{Game} to be the class of arenas for which \textit{player} has a strategy for winning the game. Note that in an arena \(A = (A, P, O, E, s, t)\), \textit{player} has a strategy to win from an element \(a\) if \textit{either} \(a \in P\) and there is some move from \(a\) so that \textit{player} still has a strategy to win after that move \textit{or} \(a \in O\) and for every move from \(a\), \textit{player} can win after that move.

(a) Give a sentence of LFP that defines the class of structures \textsc{Game}.

We say that a collection \(C\) of decision problems is \textit{closed under logarithmic space reductions} if whenever \(A \in C\) and \(B \leq_L A\) (i.e. \(B\) is reducible to \(A\) by a logarithmic-space reduction) then \(B \in C\).

The class of structures \textsc{Game} defined above is known to be \(P\)-complete under logarithmic-space reductions.

(b) Explain why this, together with (a) implies that the class of problems definable in LFP is \textit{not} closed under logarithmic-space reductions.

7. Give a sentence of LFP that defines the class of linear orders with an even number of elements.

8. The \textit{directed graph reachability problem} is the problem of deciding, given a structure \((V, E, s, t)\) where \(E\) is an arbitrary binary relation on \(V\), and \(s, t \in V\), whether \((s, t)\) is in the reflexive-transitive closure of \(E\). This problem is known to be decidable in \(NL\).

Transitive closure logic is the extension of first-order logic with an operator \(tc\) which allows us to form formulae

\[ \phi \equiv [tc_{x,y}\psi](t_1, t_2) \]

where \(x\) and \(y\) are \(k\)-tuples of variables and \(t_1\) and \(t_2\) are \(k\)-tuples of terms, for some \(k\); and all occurrences of variables \(x\) and \(y\) in \(\psi\) are bound in \(\phi\). The semantics is given by saying, if \(a\) is an interpretation for the free variables of \(\phi\), then \(A \models \phi[a]\) just in case \((t_1^{a}, t_2^{a})\) is in the reflexive-transitive closure of the binary relation defined by \(\psi(x, y)\) on \(A^k\).

(a) Show that any class of structures definable by a sentence \(\phi\), as above, where \(\psi\) is first-order, is decidable in \(NL\).

(b) Show that, if \(K\) is an isomorphism-closed class of structures in a relational signature including \(<\), such that each structure in \(K\) interprets \(<\) as a linear order and

\[ \{[A]_< \mid A \in K \} \]

is decidable in \(NL\), then there is a sentence of transitive-closure logic that defines \(K\).

9. For a binary relation \(E\) on a set \(A\), define its \textit{deterministic transitive closure} to be the set of pairs \((a, b)\) for which there are \(c_1, \ldots, c_n \in A\) such
that \(a = c_1, b = c_n\) and for each \(i < n\), \(c_{i+1}\) is the unique element of \(A\) with \((c_i, c_{i+1}) \in E\).

Let \(\text{DTC}\) denote the logic formed by extending first-order logic with an operator \(\text{dtc}\) with syntax analogous to \(\text{tc}\) above, where \([\text{dtc}_{x,y}\psi]\) defines the deterministic transitive closure of \(\psi(x,y)\).

(a) Show that every sentence of \(\text{DTC}\) defines a class of structures decidable in \(L\).

(b) Show that, if \(K\) is an isomorphism-closed class of structures in a relational signature including \(<\), such that each structure in \(K\) interprets \(<\) as a linear order and

\[
\{[A]_\prec \mid A \in K\}
\]

is decidable in \(L\), then there is a sentence of \(\text{DTC}\) that defines \(K\).

10. The structure homomorphism problem for a relational structure \(\sigma\) (with no function or constant symbols) is the problem of deciding, given two \(\sigma\)-structures \(A\) and \(B\) whether there is a homomorphism \(A \to B\).

(a) Show that if \(\sigma\) contains only unary relations, then the structure homomorphism problem for \(\sigma\) is decidable in polynomial time.

(b) Show that if \(\sigma\) contains a relation of arity 2 or more, then the the structure homomorphism problem for \(\sigma\) is \(\text{NP}\)-complete.

11. Schaefer’s theorem (Handout 5, page 12) gives six conditions under which \(\text{CSP}(B)\) is in \(P\), for \(B\) a structure on domain \(\{0,1\}\). For each of the six conditions, show that indeed any \(\text{CSP}(B)\) is in \(P\).

For the first five conditions, it is also the case that \(\text{CSP}(B)\) is definable in LFP. Prove this.

12. We saw (Handout 6, page 3) that for any \(B\), \(\text{CSP}(B)\) is definable in MSO. Write out an MSO formula for 3-SAT as defined on page 9 of Handout 5.

13. Prove that if \(\text{CSP}(B)\) has bounded width, it is definable in LFP.

14. We saw in the lecture that \(\text{CSP}(K_2)\) has width 3. Prove that \(\text{CSP}(K_3)\) does not have width 3. (Hint: Consider the graph \(K_4\)).