Exercise 1 (Limits and colimits).
1. Give an explicit description of $\omega^{op}$-limits (that is, limits of $\omega^{op}$-chains) in $\text{Set}$.\(^1\)
2. Give an explicit description of $\omega$-colimits (that is, colimits of $\omega$-chains) in $\text{Set}$.\(^2\)
3. Prove, giving details, that presheaf categories are complete and cocomplete.
4. Show that $\omega$-colimits commute with finite limits in $\text{Set}$.\(^2\)
5. Prove that a natural transformation in a presheaf category is a monomorphism (resp. an epimorphism) iff all its components are injections (resp. surjections).

Exercise 2 (Actions and coactions). Let $C$ be a small category and write $|C|$ for its set of objects. For $X \in \text{Set}^{\overline{|C|}}$ and $c \in C$, consider the following definitions:

$$\Diamond(X)(c) = \sum_{z \in C_0} X(z) \times C(c, z), \quad \Box(X)(c) = \prod_{z \in C_0} C(z, c) \Rightarrow X(z).$$

1. Respectively turn the $\Diamond$ and the $\Box$ constructions into a monad and a comonad on $\text{Set}^{\overline{|C|}}$.
2. What is the relationship between $\Diamond$ and $\Box$? Prove your claim.
3. Prove that $\Diamond\text{-Alg} \cong \hat{C} \cong \Box\text{-coAlg}$. 

Exercise 3 (Cartesian closure).
1. Prove, giving details, that presheaf categories are cartesian closed.
2. Let $\mathbb{C}$ have binary products, and consider $c \in \mathbb{C}$ and $P \in \mathbb{C}^{\text{op}}$. 
   (a) Give an explicit description of the exponential $P^{Y(c)}$ in $\mathbb{C}$.
   (b) Prove that the exponential endofunctor $(-)^{Y(c)}$ on $\mathbb{C}$ has a right adjoint.
3. Prove that the Yoneda embedding preserves limits and exponentials but need not preserve colimits.

Exercise 4 (Ends). For $P, Q \in \mathbb{C}$, prove that

$$\mathbb{C}(P, Q) \cong \int_{c \in \mathbb{C}} P(c) \Rightarrow Q(c)$$

Exercise 5 (Coends).
1. For $S \in \text{Set}$ and $H : \mathbb{C}^{\text{op}} \times \mathbb{C} \rightarrow \text{Set}$, prove that

$$\int_{c \in \mathbb{C}} S \times H(c, c) \cong S \times \int_{c \in \mathbb{C}} H(c, c)$$

\(^1\)For a further challenge, generalise this to projective limits.
\(^2\)For a further challenge, generalise this to filtered colimits.
2. Let $\mathbf{F}$ be the category of finite sets and functions between them.

For $P, Q \in \mathbf{Set}^{\mathbf{F}}$, let $P \star Q \in \mathbf{Set}^{\mathbf{F}}$ be given, for $c \in \mathbf{F}$, by

$$(P \star Q)(c) = \int_{a \in \mathbf{F}} \int_{b \in \mathbf{F}} P(a) \times Q(b) \times \mathbf{F}(a + b, c)$$

Prove that $P \star Q \cong P \times Q$.

Exercise 6 (Category of elements). Let $P$ be a presheaf in $\widehat{\mathbf{C}}$ for a small category $\mathbf{C}$.

1. Prove that the category $\int P$ of elements of $P$ and the comma category $y/P$ are isomorphic (over $\mathbf{C}$).

2. Prove that $P$ is a colimit of representable presheaves indexed by the category of elements $\int P$.

3. Prove that the slice category $\widehat{\mathbf{C}}/P$ and the presheaf category $\mathbf{dR} P$ are isomorphic.

Exercise 7 (Extensions).

1. Let $F^\# : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ be the cocontinuous extension of $F : \mathbf{A} \to \mathbf{B}$ given by

$$F^\#(P)(b) = \int_{a \in \mathbf{A}} P(a) \times F(a)(b) \quad (P \in \widehat{\mathbf{A}}, b \in \mathbf{B}^{op})$$

(a) For $y_{\mathbf{A}} : \mathbf{A} \hookrightarrow \widehat{\mathbf{A}}$ the Yoneda embedding, prove that $F \cong F^\# y_{\mathbf{A}}$ and $y_{\mathbf{A}}^\# \cong \text{Id}_{\widehat{\mathbf{A}}}$.

(b) For $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \widehat{\mathbf{C}}$, prove that $(G^\# F^\#) \cong G^\# F^\# : \widehat{\mathbf{A}} \to \widehat{\mathbf{C}}$.

2. A left Kan extension of a functor $F : \mathbf{A} \to \mathbf{B}$ along the Yoneda embedding $y_{\mathbf{A}} : \mathbf{A} \hookrightarrow \widehat{\mathbf{A}}$ consists of a functor $\text{Lan}_{\mathbf{A}}(F) : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ together with a natural transformation $\lambda_F : F \Rightarrow \text{Lan}_{\mathbf{A}}(F) y_{\mathbf{A}}$ satisfying the following universal property: for all functors $G : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ and natural transformations $\varphi : F \Rightarrow G \circ y_{\mathbf{A}}$ there exists a unique natural transformation $\gamma : \text{Lan}_{\mathbf{A}}(F) \Rightarrow G$ such that $(\gamma y_{\mathbf{A}}) \lambda_F = \varphi$.

Considering a suitable $\lambda : F \Rightarrow F^\# y_{\mathbf{A}}$, prove that $F^\#$ and $\lambda$ constitute a left Kan extension of $F$ along $y_{\mathbf{A}}$.

Exercise 8 (Essential geometric morphisms).

1. Let $\mathbf{C}$ be a small category and write $\mathbf{C}_0$ for the discrete category on its set of objects $|\mathbf{C}|$.

Give an elementary description of the essential geometric morphism

$$\begin{array}{ccc}
\widehat{\mathbf{C}_0} & \downarrow i & \downarrow \gamma \\
\widehat{\mathbf{C}} & & \\
\end{array}$$

induced by the inclusion $\mathbf{C}_0 \to \mathbf{C}$.

2. Let $\mathbf{C}$ have binary products. For $c \in \mathbf{C}$, prove that $((-) \times c)_! \cong (-) \times y(c)$.

3. Let $f : \mathbf{A} \to \mathbf{B}$ be a functor between small categories.

(a) Prove that $f$ is an embedding iff so is $f_! : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$.

(b) Prove that if $f$ is an embedding then so is $f_* : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$.

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\textsuperscript{3}Reconsider Exercise 3(2) in this light.
Exercise 9 (Free algebras). 1. Prove Lambek’s lemma and state its dual.

2. Let $F$ be an endofunctor on a category with initial object $0$. Prove that if the $\omega$-chain below (obtained by iterating $F$ at $0$)

$$
\begin{array}{c}
0 \\
F^0 \\
F^1 \\
\vdots \\
F^{n+1} \\
\end{array}
\xrightarrow{F}
\begin{array}{c}
F^1 \\
F^2 \\
\vdots \\
F^n \\
\end{array}
$$

has a colimit and $F$ preserves it then $F$ has an initial algebra. State the dual of this result.

3. For an $\omega$-cocontinuous endofunctor (that is, one that preserves $\omega$-colimits) $F$ on a category $\mathcal{C}$ with finite coproducts and $\omega$-colimits, prove that the forgetful functor $F\text{-alg} \to \mathcal{C}$ has a left adjoint. State the dual of this result.

4. State the dual of this result.

Exercise 10 (Abstract syntax and variable binding). Consider the syntax of the lambda calculus (up to alpha equivalence) given by the following rules, where $\Gamma$ ranges over contexts (namely, finite sets from a fixed countably infinite set of variables):

$$
\begin{array}{c}
x \in \Gamma \\
\Gamma \vdash x \\
\Gamma \vdash t_1 \\
\Gamma \vdash t_2 \\
\Gamma \vdash t_1(t_2) \\
\Gamma, x \vdash t \\
\end{array}
$$

here $\Gamma, x$ denotes $\Gamma \cup \{x\}$ under the assumption $x \notin \Gamma$.

Let $\mathcal{C}$ be the category with objects contexts and with morphisms functions between them.

1. Equip the following two indexed families

$$
\begin{array}{c}
V(\Gamma) = \Gamma \\
\text{and} \\
L(\Gamma) = \{ t \mid \Gamma \vdash t \}. \\
(\Gamma \in \mathcal{C})
\end{array}
$$

with actions making them into presheaves $V$ and $L$ in $\text{Set}^\mathcal{C}$.

2. For $\Sigma$ the endofunctor on $\text{Set}^\mathcal{C}$ given by $\Sigma(X) = X \times X + X^V$, prove that $L$ has the structure of a free $\Sigma$-algebra on $V$.

Exercise 11 (Day convolution and Joyal species). Let $\mathcal{B}$ be the category with objects finite sets and with morphisms bijective functions between them.

Devise syntax for terms $t$ and rules for judgements $\Gamma \vdash t$ with contexts $\Gamma \in \mathcal{B}$ such that the indexed family

$$
T(\Gamma) = \{ t \mid \Gamma \vdash t \} \\
(\Gamma \in \mathcal{B})
$$

can be equipped with an action making it into a Joyal species

$$
T \in \text{Set}^\mathcal{B}
$$

satisfying the isomorphism

$$
T \cong N + T \otimes T + N \rightarrow T
$$

where $N$ denotes $y\{x\}$, and $\otimes$ and $\rightarrow$ arise by Day convolution:

$$
(P \otimes Q)(\Gamma) = \int_{\Gamma_1, \Gamma_2 \in \mathcal{B}} P(\Gamma_1) \times Q(\Gamma_2) \times \mathcal{B}(\Gamma_1 \uplus \Gamma_2, \Gamma)
$$

$$
(P \rightarrow Q)(\Gamma) = \int_{\Delta \in \mathcal{B}} P(\Delta) \Rightarrow Q(\Gamma \uplus \Delta)
$$

Show that (1) and (2) hold.

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4For a further challenge, generalise this to $\mathcal{C}$ having binary coproducts.