Advanced Topics in Category Theory 2022-23

Exercises for Part II

(Version 1)

Exercise 1 (Limits and colimits)**.**

1. *Give an explicit description of ω^{op}-limits (that is, limits of ω^{op}-chains) in* **Set**.[1]
2. *Give an explicit description of ω-colimits (that is, colimits of ω-chains) in* **Set**.[2]
3. *Prove, giving details, that presheaf categories are complete and cocomplete.*
4. *Show that ω-colimits commute with finite limits in* **Set**.[2]
5. *Prove that a natural transformation in a presheaf category is a monomorphism (resp. an epimorphism) iff all its components are injections (resp. surjections).*

Exercise 2 (Actions and coactions)**.** *Let $\mathbb{C}$ be a small category and write $|\mathbb{C}|$ for its set of objects. For $X \in \mathbf{Set}^{|\mathbb{C}|}$ and $c \in \mathbb{C}$, consider the following definitions:*

$$\Diamond(X)(c) = \sum_{z \in \mathbb{C}_0} X(z) \times \mathbb{C}(c, z) \quad , \qquad \Box(X)(c) = \prod_{z \in \mathbb{C}_0} \mathbb{C}(z, c) \Rightarrow X(z) \quad .$$

1. *Respectively turn the $\Diamond$ and the $\Box$ constructions into a monad and a comonad on $\mathbf{Set}^{|\mathbb{C}|}$.*
2. *What is the relationship between $\Diamond$ and $\Box$? Prove your claim.*
3. *Prove that $\Diamond\text{-}\mathbf{Alg} \cong \widehat{\mathbb{C}} \cong \Box\text{-}\mathbf{coAlg}$.*

Exercise 3 (Cartesian closure)**.**

1. *Prove, giving details, that presheaf categories are cartesian closed.*
2. *Let $\mathbb{C}$ have binary products, and consider $c \in \mathbb{C}$ and $P \in \widehat{\mathbb{C}}$.*
   (a) *Give an explicit description of the exponential $P^{\boldsymbol{y}(c)}$ in $\widehat{\mathbb{C}}$.*
   (b) *Prove that the exponential endofunctor $(-)^{\boldsymbol{y}(c)}$ on $\widehat{\mathbb{C}}$ has a right adjoint.*
3. *Prove that the Yoneda embedding preserves limits and exponentials but need not preserve colimits.*

Exercise 4 (Ends)**.** *For $P, Q \in \widehat{\mathbb{C}}$, prove that*

$$\widehat{\mathbb{C}}(P, Q) \;\cong\; \int_{c \in \mathbb{C}} P(c) \Rightarrow Q(c)$$

Exercise 5 (Coends)**.**

1. *For $S \in \mathbf{Set}$ and $H : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$, prove that*

$$\int^{c \in \mathbb{C}} S \times H(c, c) \;\cong\; S \times \int^{c \in \mathbb{C}} H(c, c)$$

[1] For a further challenge, generalise this to projective limits.
[2] For a further challenge, generalise this to filtered colimits.

2. *Let $\mathbf{F}$ be the category of finite sets and functions between them.*

   For $P, Q \in \mathbf{Set}^{\mathbf{F}}$, let $P \star Q \in \mathbf{Set}^{\mathbf{F}}$ be given, for $c \in \mathbf{F}$, by

$$(P \star Q)(c) \;=\; \int^{a \in \mathbf{F}} \int^{b \in \mathbf{F}} P(a) \times Q(b) \times \mathbf{F}(a+b, c)$$

   Prove that $P \star Q \;\cong\; P \times Q$.

Exercise 6 (Category of elements)**.** *Let P be a presheaf in $\widehat{\mathbb{C}}$ for a small category $\mathbb{C}$.*

1. *Prove that the category $\int P$ of elements of P and the comma category $\boldsymbol{y}/P$ are isomorphic (over $\mathbb{C}$).*

2. *Prove that P is a colimit of representable presheaves indexed by the category of elements $\int P$.*

3. *Prove that the slice category $\widehat{\mathbb{C}}/P$ and the presheaf category $\widehat{\int P}$ are isomorphic.*

Exercise 7 (Extensions)**.**

1. *Let $F^{\#} : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ be the cocontinuous extension of $F : \mathbb{A} \to \widehat{\mathbb{B}}$ given by*

$$F^{\#}(P)(b) = \int^{a \in \mathbb{A}} P(a) \times F(a)(b) \qquad\qquad (P \in \widehat{\mathbb{A}}, b \in \mathbb{B}^{\mathrm{op}})$$

   (a) *For $\boldsymbol{y}_{\mathbb{A}} : \mathbb{A} \hookrightarrow \widehat{\mathbb{A}}$ the Yoneda embedding, prove that $F \cong F^{\#}\, \boldsymbol{y}_{\mathbb{A}}$ and ${\boldsymbol{y}_{\mathbb{A}}}^{\#} \cong \mathrm{Id}_{\widehat{\mathbb{A}}}$.*

   (b) *For $F : \mathbb{A} \to \widehat{\mathbb{B}}$ and $G : \mathbb{B} \to \widehat{\mathbb{C}}$, prove that $(G^{\#}\, F)^{\#} \;\cong\; G^{\#}\, F^{\#} : \widehat{\mathbb{A}} \to \widehat{\mathbb{C}}$.*

2. *A left Kan extension of a functor $F : \mathbb{A} \to \widehat{\mathbb{B}}$ along the Yoneda embedding $\boldsymbol{y}_{\mathbb{A}} : \mathbb{A} \hookrightarrow \widehat{\mathbb{A}}$ consists of a functor $\mathrm{Lan}_{\mathbb{A}}(F) : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ together with a natural transformation $\lambda_F : F \Longrightarrow \mathrm{Lan}_{\mathbb{A}}(F)\, \boldsymbol{y}_{\mathbb{A}}$ satisfying the following universal property: for all functors $G : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$ and natural transformations $\varphi : F \Longrightarrow G \circ \boldsymbol{y}_{\mathbb{A}}$ there exists a unique natural transformation $\gamma : \mathrm{Lan}_{\mathbb{A}}(F) \Longrightarrow G$ such that $(\gamma\, \boldsymbol{y}_{\mathbb{A}})\, \lambda_F = \varphi$.*

   Considering a suitable $\lambda : F \Longrightarrow F^{\#}\, \boldsymbol{y}_{\mathbb{A}}$, prove that $F^{\#}$ and λ constitute a left Kan extension of F along $\boldsymbol{y}_{\mathbb{A}}$.

Exercise 8 (Essential geometric morphisms)**.**

1. *Let $\mathbb{C}$ be a small category and write $\mathbb{C}_0$ for the discrete category on its set of objects $|\,\mathbb{C}\,|$. Give an elementary description of the essential geometric morphism*

$$\begin{tikzcd} \widehat{\mathbb{C}_0} \arrow[d, shift right=6] \arrow[d, phantom, shift right=3, "\dashv"] \arrow[d, phantom, shift left=3, "\dashv"] \arrow[d, shift left=6] \\ \widehat{\mathbb{C}} \arrow[u] \end{tikzcd}$$

   induced by the inclusion $\mathbb{C}_0 \to \mathbb{C}$.

2. *Let $\mathbb{C}$ have binary products. For $c \in \mathbb{C}$, prove that $\big((-) \times c\big)_! \cong (-) \times \boldsymbol{y}(c)$.*[3]

3. *Let $f : \mathbb{A} \to \mathbb{B}$ be a functor between small categories.*

   (a) *Prove that f is an embedding iff so is $f_! : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$.*

   (b) *Prove that if f is an embedding then so is $f_* : \widehat{\mathbb{A}} \to \widehat{\mathbb{B}}$.*

[3]Reconsider Exercise 3(2) in this light.

Exercise 9 (Free algebras). *1. Prove Lambek's lemma and state its dual.*

2. *Let F be an endofunctor on a category with initial object 0. Prove that if the ω-chain below (obtained by interating F at 0)*

$$0 \xrightarrow{!} F0 \xrightarrow{F!} F^2 0 \longrightarrow \cdots \longrightarrow F^n 0 \xrightarrow{F^n!} F^{n+1} 0 \longrightarrow \cdots$$

has a colimit and F preserves it then F has an initial algebra. State the dual of this result.

3. *For an ω-cocontinuous endofunctor (that is, one that preserves ω-colimits) F on a category $\mathcal{C}$ with finite coproducts and ω-colimits, prove that the forgetful functor F-$\mathbf{alg} \to \mathcal{C}$ has a left adjoint.*[4] *State the dual of this result.*

4. *For an endofunctor F on a category $\mathcal{C}$, prove that if the forgetful functor F-$\mathbf{alg} \to \mathcal{C}$ has a left adjoint then it is monadic. State the dual of this result.*

Exercise 10 (Abstract syntax and variable binding). *Consider the syntax of the lambda calculus (up to alpha equivalence) given by the following rules, where Γ ranges over contexts (namely, finite sets from a fixed countably infinite set of variables):*

$$\frac{x \in \Gamma}{\Gamma \vdash x} \qquad\qquad \frac{\Gamma \vdash t_1 \quad \Gamma \vdash t_2}{\Gamma \vdash t_1(t_2)} \qquad\qquad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t}$$

here Γ, x denotes $\Gamma \cup \{x\}$ under the assumption $x \notin \Gamma$.

Let $\mathbf{C}$ be the category with objects contexts and with morphisms functions between them.

1. *Equip the following two indexed families*

$$V(\Gamma) = \Gamma \qquad \textit{and} \qquad L(\Gamma) = \{\, t \mid \Gamma \vdash t \,\}. \qquad\qquad (\Gamma \in \mathbf{C})$$

with actions making them into presheaves V and L in $\mathbf{Set}^{\mathbf{C}}$.

2. *For Σ the endofunctor on $\mathbf{Set}^{\mathbf{C}}$ given by $\Sigma(X) = X \times X + X^V$, prove that L has the structure of a free Σ-algebra on V.*

Exercise 11 (Day convolution and Joyal species). *Let $\mathbf{B}$ be the category with objects finite sets and with morphisms bijective functions between them.*

Devise syntax for terms t and rules for judgements $\Gamma \vdash t$ with contexts $\Gamma \in \mathbf{B}$ such that the indexed family

$$T(\Gamma) = \{\, t \mid \Gamma \vdash t \,\} \qquad\qquad (\Gamma \in \mathbf{B})$$

can be equipped with an action making it into a Joyal species

$$T \in \mathbf{Set}^{\mathbf{B}} \tag{1}$$

satisfying the isomorphism

$$T \;\cong\; N + T \otimes T + N \multimap T \tag{2}$$

where N denotes $\boldsymbol{y}\{x\}$, and $\otimes$ and $\multimap$ arise by Day convolution:

$$(P \otimes Q)(\Gamma) \;=\; \int^{\Gamma_1,\Gamma_2 \in \mathbf{B}} P(\Gamma_1) \times Q(\Gamma_2) \times \mathbf{B}(\Gamma_1 \uplus \Gamma_2, \Gamma)$$

$$(P \multimap Q)(\Gamma) \;=\; \int_{\Delta \in \mathbf{B}} P(\Delta) \Rightarrow Q(\Gamma \uplus \Delta)$$

Show that (1) and (2) hold.

[4] For a further challenge, generalise this to $\mathcal{C}$ having binary coproducts.