# Ordered pairing

### Notation:

(a,b) or  $\langle a,b \rangle$ 

### Fundamental property:

 $(a,b) = (x,y) \implies a = x \land b = y$ 

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### A construction:

For every pair a and b, three applications of the pairing axiom provide the set

$$\langle a,b\rangle = \{ \{a\}, \{a,b\} \}$$

which defines an ordered pairing of a and b.

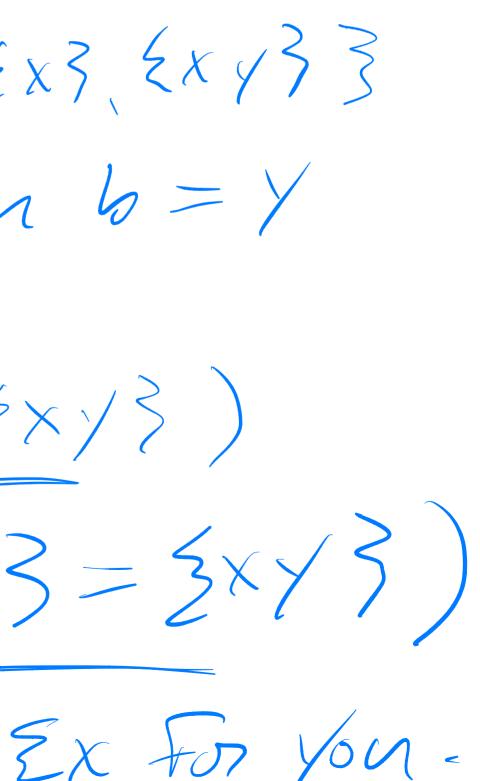
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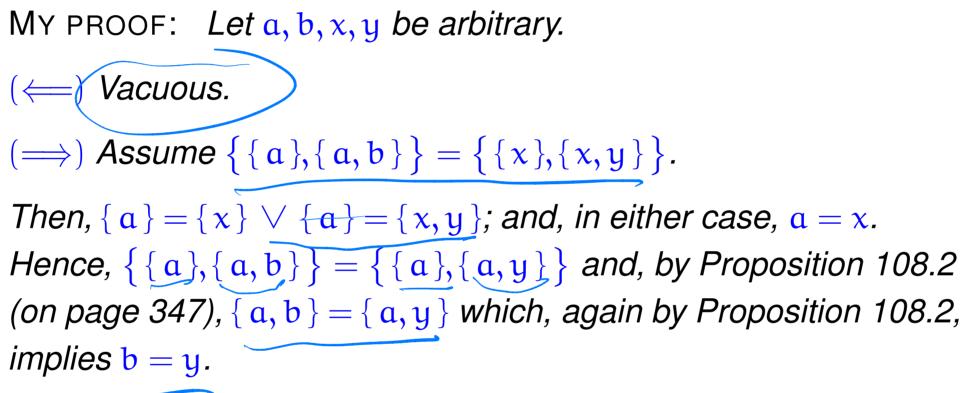
Proposition 109 (Fundamental property of ordered pairing) For all a, b, x, y,

 $\langle a, b \rangle = \langle x, y \rangle \iff (a = x \land b = y)$ .

YOUR PROOF:

\$ 29), 29,633 = { Ex3, 2×133  $\rightarrow \alpha = \chi \land b = Y$ E Casy  $\left(\left(\frac{\xi u}{3} = \frac{\xi}{2} \times \frac{3}{3} \vee \frac{\xi u}{3} = \frac{\xi}{2} \times \frac{1}{3}\right)$  $\Lambda(29,63=283\sqrt{29,63}=2843)$ R 64.5





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# Products

The product  $A \times B$  of two sets A and B is the set

$$A \times B \neq \{x \mid \exists a \in A, b \in B. x = (a, b)\}$$

where

$$\begin{pmatrix} \forall a_1, a_2 \in A, b_1, b_2 \in B. \\ (a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \land b_1 = b \end{pmatrix}$$

Thus,

$$\forall x \in A \times B. \exists ! a \in A. \exists ! b \in B. x = (a, b)$$

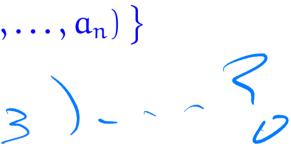
**5**<sub>2</sub>) .

•

More generally, for a fixed natural number n and sets  $A_1, \ldots, A_n$ , we have  $\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n$ =  $\{x \mid \exists a_1 \in A_1, \dots, a_n \in A_n, x = (a_1, \dots, a_n)\}$  $= A_n \cdot \left( \left( a_1, a_2 \right) a_3 \right) - \left( a_$ where  $\forall a_1, a'_1 \in A_1, \ldots, a_n, a'_n \in A_n$  $(a_1,\ldots,a_n)=(a'_1,\ldots,a'_n)\iff (a_1=a'_1\wedge\cdots\wedge a_n=a'_n)$ .

Notation 110 For a natural number n and a set A, one typically writes  $A^n$  for  $\prod_{i=1}^n A$ .





- **NB** Cunningly enough, the definition is such that  $\prod_{i=1}^{0} A_i = \{()\}$ .

### Pattern-matching notation

**Example:** The subset of ordered pairs from a set A with equal components is formally

 $\{x \in A \times A \mid \exists a_1 \in A. \exists a_2 \in A. x = (\underline{a_1, a_2}) \land a_1 = a_2\}$ 

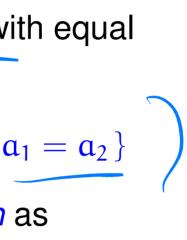
but often abbreviated using *pattern-matching notation* as

 $\{(a_1, a_2) \in A \times A \mid a_1 = a_2\}$ 

**Notation:** For a property P(a, b) with a ranging over a set A and b ranging over a set **B**,

$$\{(a,b) \in A \times B \mid \underline{P(a,b)}\}$$

abbreviates



Proposition 111 For all finite sets A and B,

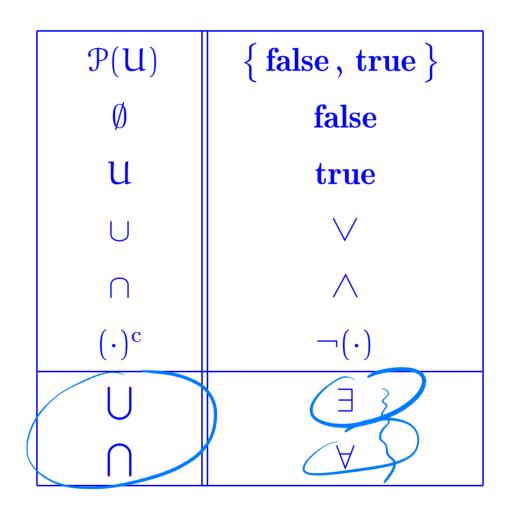
 $\#(A \times B) = \#A \cdot \#B$ .

PROOF IDEA <sup>a</sup> :

 $A = \frac{3}{2}a_{1} - \frac{3}{2}a_{2}x^{2} + \frac{3}{2}a_{3}x^{2} + \frac{3}{$ (ax, 6) - - - - - - (ax, b;) ]

<sup>a</sup>See Theorem 162.2 on page 439. — **356** —

# Sets and logic



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### Big unions

### **Example:**



Consider the family of sets

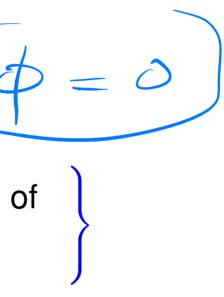
 $\mathcal{T} = \left\{ \begin{array}{c} T \subseteq \underline{[5]} \\ T \end{bmatrix} \right. \begin{array}{c} \text{the sum of the elements of} \\ T \text{ is less than or equal 2} \end{array} \right\}$ 

 $= \{\emptyset, \{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$ 

• The *big union* of the family  $\mathcal{T}$  is the set  $\bigcup \mathcal{T}$  given by the union of the sets in T:

 $n \in \bigcup \mathfrak{T} \iff \exists T \in \mathfrak{T}. n \in T$ .

Hence,  $\bigcup \mathcal{T} = \{0, 1, 2\}.$ -358 -



**Definition 112** Let U be a set. For a collection of sets  $\mathcal{F} \in \mathcal{P}(\mathcal{P}(U))$ , we let the big union (relative to U) be defined as

 $\underbrace{\bigcup \mathcal{F}}_{-} = \left\{ \underbrace{x \in U \mid \exists A \in \mathcal{F}. x \in A}_{\swarrow} \right\} \in \underbrace{\mathcal{P}(U)}_{\checkmark} .$ 

**Btw** To get some intuition behind this definition, it might be useful to compare the construction with the ML function

flatten : 'a list list -> 'a list

associated with the ML list datatype constructor.

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## Examples:

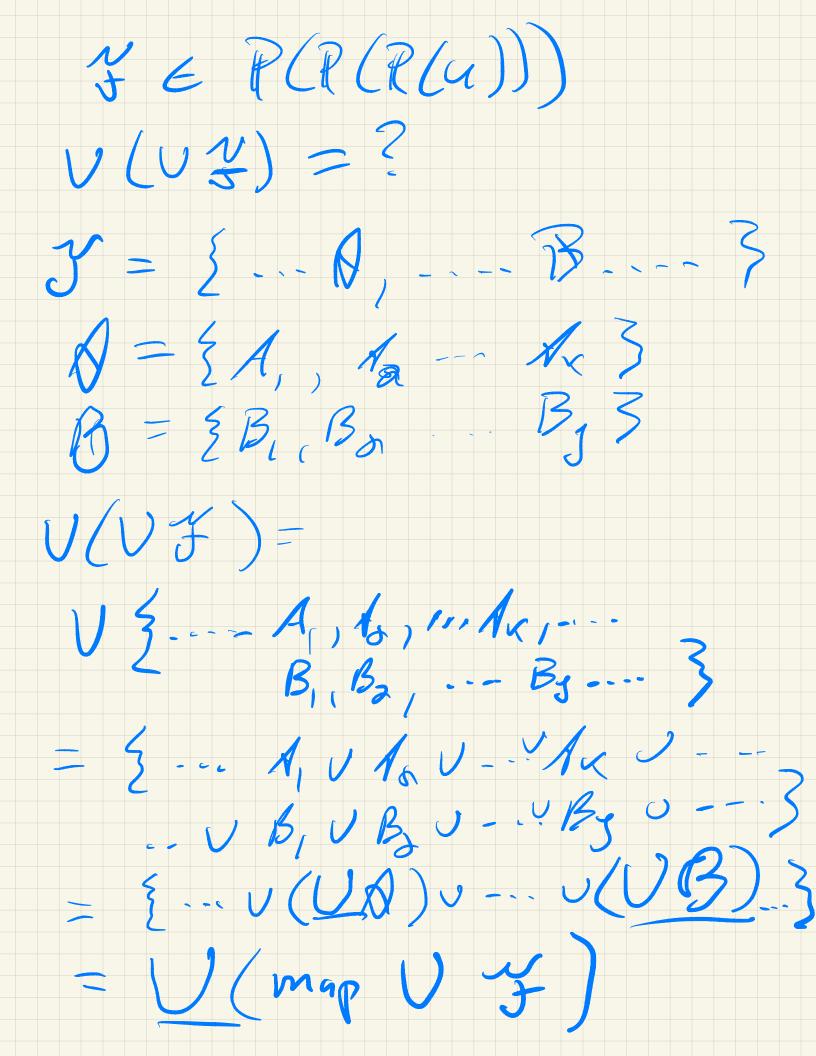
1. For  $A, A_1, A_2 \in \mathcal{P}(U)$ ,

$$\bigcup \emptyset = \emptyset$$
$$\bigcup \{A\} = A$$
$$\bigcup \{A_1, A_2\} = A_1 \cup A_2$$
$$\bigcup \{A, A_1, A_2\} = A \cup A_1 \cup A_2$$

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2. For  $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$ , let us introduce the notation  $\left\{ \begin{array}{c} \bigcup \mathcal{A} \in \mathcal{P}(\mathbf{U}) & \mathcal{A} \in \mathcal{F} \\ \end{array} \right\}$  et  $\begin{array}{c} map & \bigcup & \mathcal{F} \\ \end{array}$ for the set  $\left\{ X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}. X = \bigcup \mathcal{A} \right\} \in \mathcal{P}(\mathcal{P}(U))$ noticing that this is justified by the fact that, for all  $x \in U$ ,  $\begin{array}{l} \begin{array}{l} \begin{array}{l} x \in \bigcup \left\{ X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}. X = \bigcup \mathcal{A} \right\} \\ \end{array} \\ \begin{array}{l} \longleftrightarrow \end{array} \exists X \in \mathcal{P}(U). \exists \mathcal{A} \in \mathcal{F}. X = \bigcup \mathcal{A} \land x \in X \\ \end{array} \\ \begin{array}{l} \longleftrightarrow \end{array} \exists \mathcal{A} \in \mathcal{F}. x \in \bigcup \mathcal{A} \end{array}$ 

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We then have the following associativity law:

**Proposition 113** For all  $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{U})))$ ,

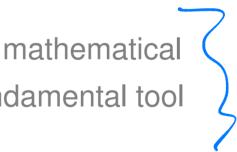
 $\underbrace{\bigcup(\bigcup\mathfrak{F})}_{}=\bigcup\left\{\left.\bigcup\mathcal{A}\,\in\,\mathbb{P}(u)\;\right|\;\mathcal{A}\in\,\mathfrak{F}\;\right\}\;\in\,\mathbb{P}(u)\;\;.$ 

**Btw** In trying to understand this statement, ponder about the following analogous identity for the ML list datatype constructor: for all F : 'a list list list,

flatten (flatten F)

= flatten ( map flatten F ) : 'a list

The above two identities are the *associativity law* of a mathematical structure known as a *monad*, which has become a fundamental tool in functional programming.



MY PROOF: For  $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{U})))$  and  $x \in \mathcal{U}$ , one calculates that:

> $\mathbf{x} \in \bigcup (\bigcup \mathcal{F})$  $\iff \exists X \in \bigcup \mathcal{F}. \ x \in X$  $\iff \exists \mathcal{A} \in \mathcal{F}. \exists X \in \mathcal{A}. x \in X$  $\iff \exists \mathcal{A} \in \mathcal{F}. \ x \in \bigcup \mathcal{A}$  $\iff x \in \bigcup \left\{ \bigcup \mathcal{A} \in \mathcal{P}(U) \mid \mathcal{A} \in \mathcal{F} \right\}$

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### **Big** intersections

### **Example:**

Consider the family of sets

$$\mathcal{S} = \begin{cases} S \subseteq [5] \end{cases}$$

▶ The *big intersection* of the family \$ is the set  $\bigcirc \$$  given by the intersection of the sets in S:

 $n \in \bigcap S \iff \forall S \in S. n \in S$ .

Hence,  $\bigcup S = \{2\}$ .

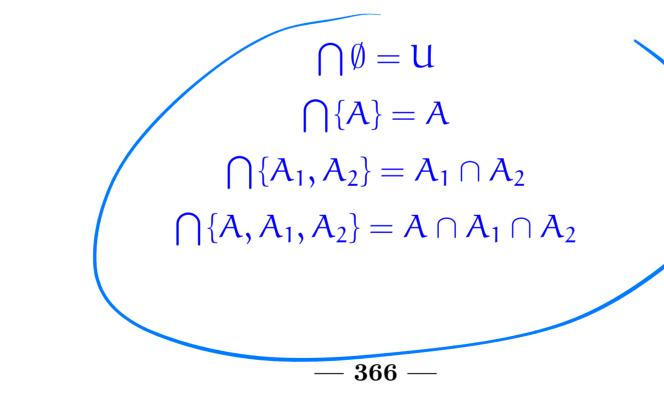
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 $= \left\{ S \subseteq [5] \middle| \begin{array}{c} \text{the sum of the elements of} \\ S \text{ is less than or equal } 6 \end{array} \right\}$  $= \left\{ \{2,4\}, \{0,2,4\}, \{1,2,3\} \} \right\}$ 

**Definition 114** Let U be a set. For a collection of sets  $\mathfrak{F} \subseteq \mathfrak{P}(U)$ , we let the big intersection (relative to U) be defined as

$$\bigcap \mathcal{F} = \left\{ x \in U \mid \forall A \notin \mathcal{F}. x \in A \right\}$$

**Examples:** For  $A, A_1, A_2 \in \mathcal{P}(U)$ ,



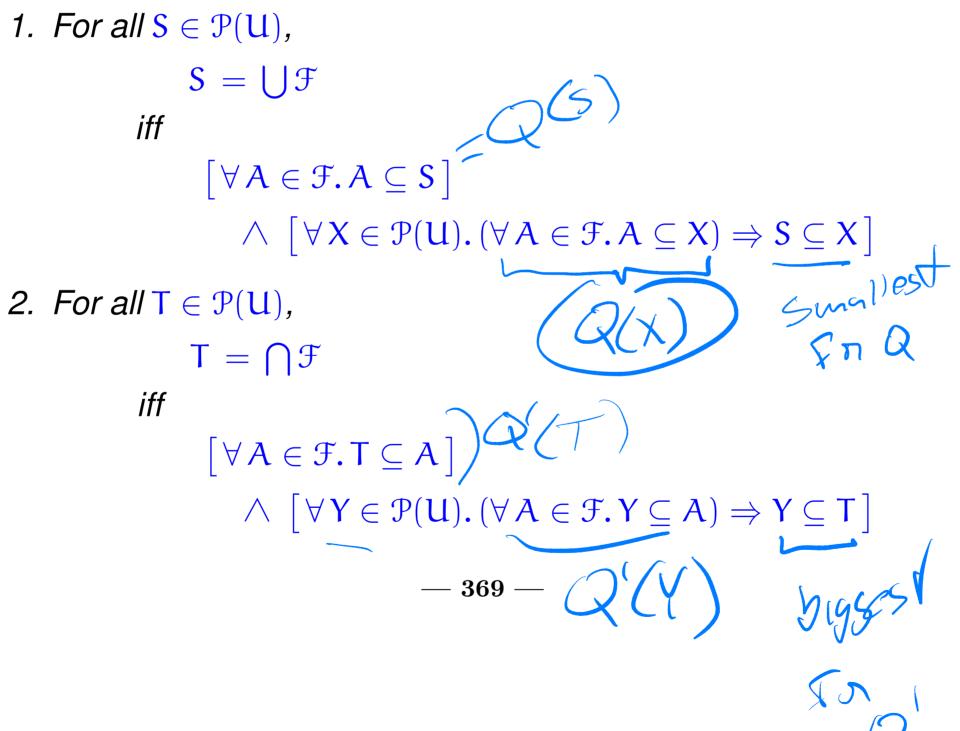


# Theorem 115 Let $\mathcal{F} = \left\{ S \subseteq \mathbb{R} \mid (0 \in S) \land (\forall x \in \mathbb{R}. x \in S \implies (x+1) \in S) \right\}.$ Then, (i) $\mathbb{N} \in \mathcal{F}$ and (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$ . Hence, $\bigcap \mathcal{F} = \mathbb{N}$ . This result is typically interpreted as stating that: NB

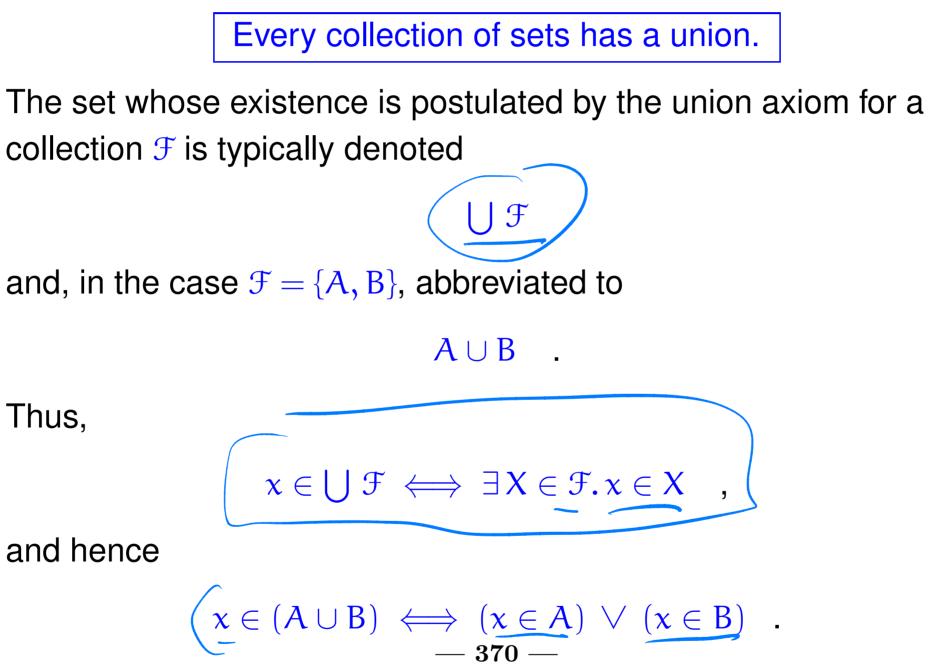
 $\begin{cases} \mathbb{N} \text{ is the least set of numbers containing 0 and closed under successors.} \\ \mathbb{P} \cap \mathbb{P} \cdot \mathbb{N} \oplus \mathbb{P} \cdot \mathbb{N} \oplus \mathbb{P} \\ \mathbb{P} \cap \mathbb{P} \cdot \mathbb{P} \cap \mathbb{P} \\ \mathbb{P} \cap \mathbb{P} \cdot \mathbb{P} \cap \mathbb{P} \\ \mathbb{P} \cap \mathbb{P} \cdot \mathbb{P} \cap \mathbb{P} \\ \mathbb{P} \cap \mathbb{P} \cap \mathbb{P} \cap \mathbb{P} \cap \mathbb{P} \\ \mathbb{P} \cap \mathbb{P} \cap \mathbb{P} \cap \mathbb{P} \cap \mathbb{P} \\ \mathbb{P} \cap \mathbb$ 



**Proposition 116** Let U be a set and let  $\mathcal{F} \subseteq \mathcal{P}(U)$  be a family of subsets of U.



### Union axiom

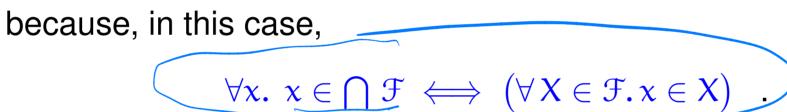


Using the separation and union axioms, for every collection  $\mathcal{F}$ , consider the set

 $\{x \in \bigcup \mathcal{F} \mid \forall X \in \mathcal{F}. x \in X\}$ .

For *non-empty*  $\mathcal{F}$  this set is denoted

 $\bigcap \mathcal{F}$ 



In particular, for  $\mathcal{F} = \{A, B\}$ , this is abbreviated to

 $A \cap B$ 

with defining property

 $\forall x. \ x \in (A \cap B) \iff (x \in A) \land (x \in B) \ .$ -371 -

