Powerset axiom

$\boxed{\text{For any set, there is a set consisting of all its subsets.}}$

$$\mathcal{P}(U)$$

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U \quad .$$

$$\mathcal{U} \longmapsto \mathcal{P}(\mathcal{U}) \longmapsto \mathcal{P}(\mathcal{P}\mathcal{U}) \longmapsto \mathcal{P}(\mathcal{P}(\mathcal{P}\mathcal{U})) \longmapsto \ldots$$

NB: The powerset construction can be iterated. In particular,

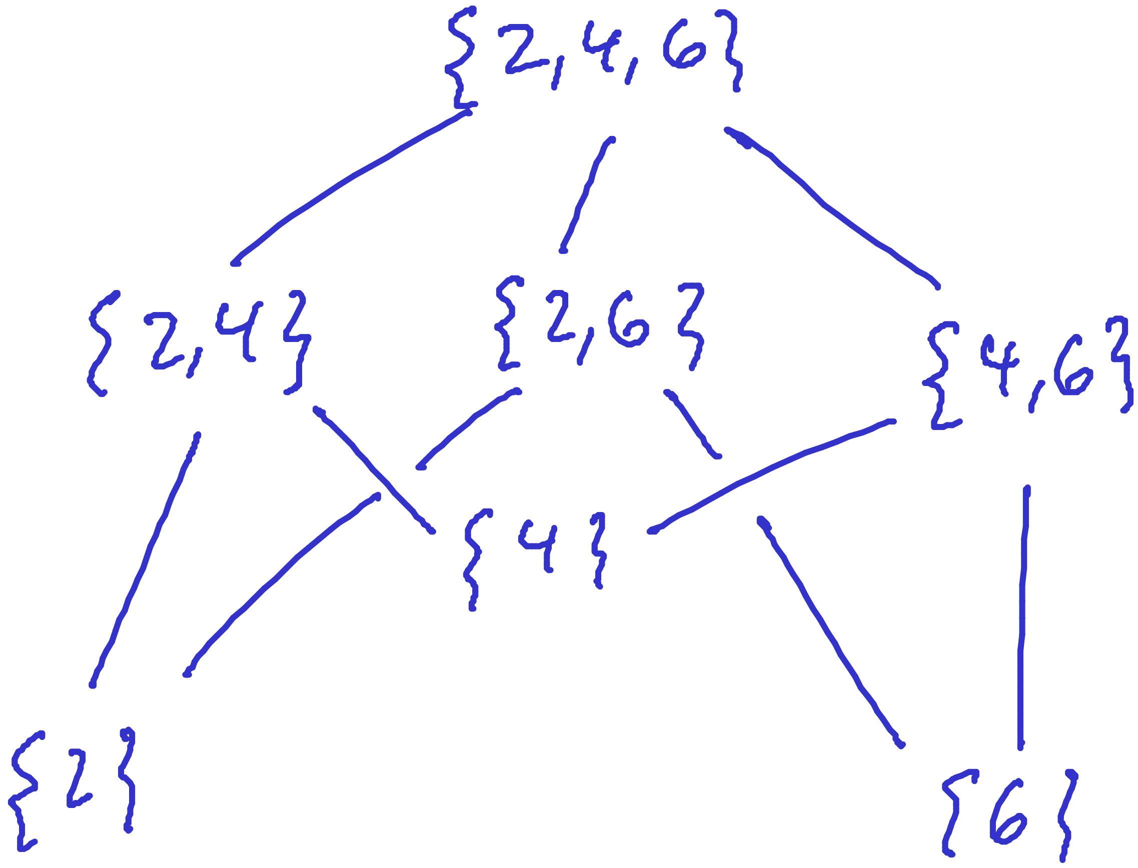
$$\mathcal{F} \in \mathcal{P}\big(\mathcal{P}(\mathcal{U})\big) \iff \mathcal{F} \subseteq \mathcal{P}(\mathcal{U}) \;\; ;$$

that is, $\mathcal{F}$ is a set of subsets of $\mathcal{U}$, sometimes referred to as a *family*.

Example: The family $\mathcal{E} \subseteq \mathcal{P}\big(\,[5]\,\big)$ consisting of the non-empty subsets of $[5] = \{\,0,1,2,3,4\,\}$ whose elements are even is

$$\mathcal{E} \;=\; \big\{\,\{0\},\{2\},\{4\},\{0,2\},\{0,4\},\{2,4\},\{0,2,4\}\,\big\}\;\; .$$

Hasse diagrams



Proposition 104 *For all finite sets* U,

$$\# \mathcal{P}(U) = 2^{\# U} \quad .$$

PROOF IDEA: $U = \{u_1, u_2, \ldots, u_n\}$ $n \in \mathbb{N}$.

Represent subsets of U:

$S \subseteq U$ $\quad \frac{(u_1 \in S)}{u_1} \quad \frac{(u_2 \in S)}{u_2} \quad \frac{\ }{u_3} \quad \ldots \quad \frac{(u_n \in S)}{u_n}$

Examples: $U \subseteq U$ $\quad \frac{\text{true}}{u_1} \quad \frac{\text{true}}{u_2} \quad \ldots \quad \frac{\text{true}}{u_n}$

$\emptyset \subseteq U$ $\quad \frac{\text{false}}{u_1} \quad \frac{\text{false}}{u_2} \quad \ldots \quad \frac{\text{false}}{u_n}$

The cardinality of

$$P(\mathcal{U})$$

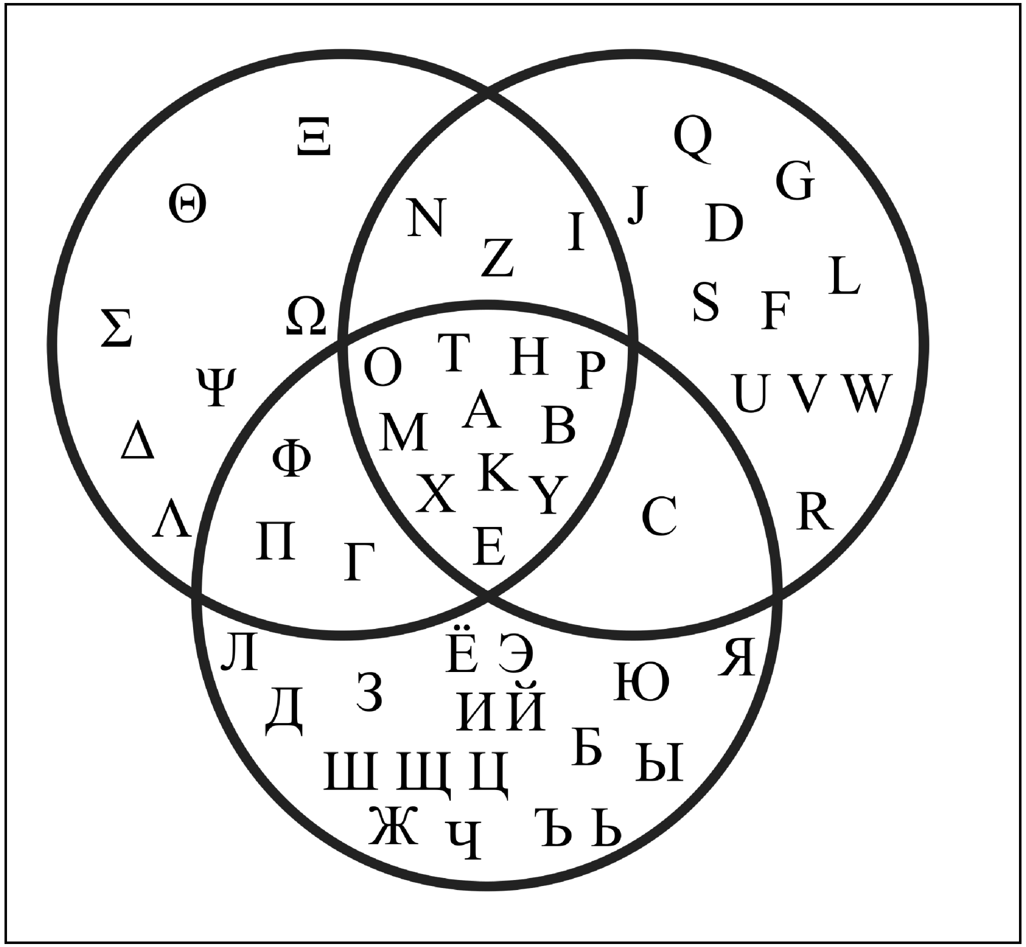
is the cardinality of the sequences of $\{true, false\}$ of length $\#\mathcal{U}$.

Example

$$P[3] = \{ \emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\} \}$$

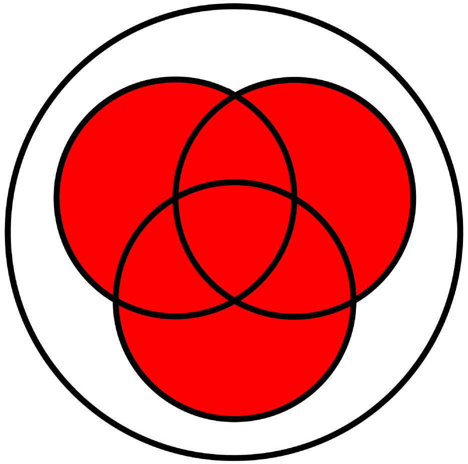
(false, false, false)
(true, false, false)
(false, true, false)
(false, false, true)
⋮
(true, true, true)

Venn diagrams[a]

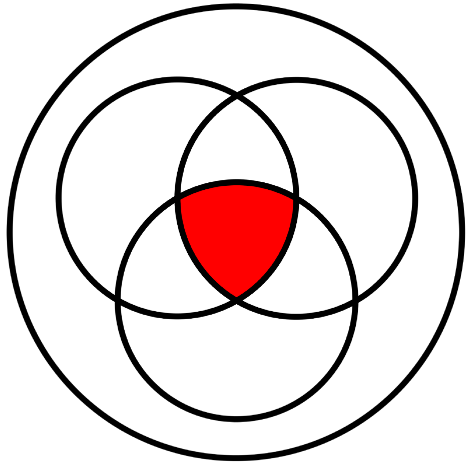


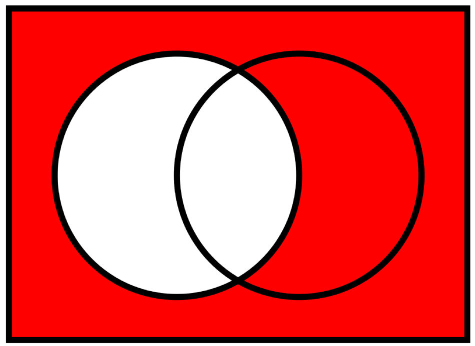
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

false true

The powerset Boolean algebra

$$(\ \mathcal{P}(U)\ ,\ \emptyset\ ,\ U\ ,\ \cup\ ,\ \cap\ ,\ (\cdot)^c\ )$$

For all $A, B \in \mathcal{P}(U)$,

$$A \cup B = \{x \in U \mid x \in A \vee x \in B\} \in \mathcal{P}(U)$$

$$A \cap B = \{x \in U \mid x \in A \wedge x \in B\} \in \mathcal{P}(U)$$

$$A^c = \{x \in U \mid \neg(x \in A)\} \in \mathcal{P}(U)$$

$\emptyset = \{x \in U \mid \text{false}\}$ $U = \{x \in U \mid \text{true}\}$.

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathcal{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A \ = \ A \ = \ \mathcal{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \ , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) \ = \ A \ = \ A \cap (A \cup B)$$

$A \cup (A \cap B) = A$.

RTP $A \cup (A \cap B) \subseteq A$

NB: $X \subseteq Y \Rightarrow A \cup X \subseteq A \cup Y$

Assume $X \subseteq Y \Leftrightarrow \boxed{\forall x \in X .\, x \in Y}$

RTP: $A \cup X \subseteq A \cup Y$

That is, $\forall z \in A \cup X .\ z \in A \cup Y$

equiv. Assume $z \in A \cup X \Leftrightarrow \boxed{(z \in A \vee z \in X)}$

RTP: $z \in A \cup Y \Leftrightarrow (z \in A \vee z \in Y)$

$A \subseteq A \cup (A \cap B)$

NB For all sets X,

$A \subseteq A \cup X$

Since $z \in A \vee z \in X$

Consider each case:

(1) $\cancel{z \in A}$ we are done.

(2) $z \in X$.

Then $z \in Y$ and we are done. $\boxtimes$

$A \cup (A \cap B) \subseteq A$

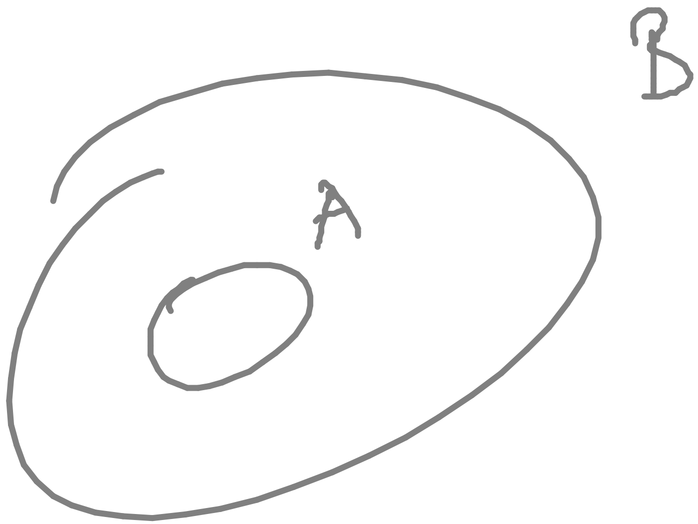
Know $A \cap B \subseteq A \Rightarrow A \cup (A \cap B) \subseteq A \cup A = A$

- The complement operation $(\cdot)^{\mathrm{c}}$ satisfies complementation laws.

$$A \cup A^{\mathrm{c}} = \mathcal{U} \ , \quad A \cap A^{\mathrm{c}} = \emptyset$$

NB:

$$\begin{cases} A \subseteq B \iff A \cup B = B \\ A \subseteq B \iff A \cap B = A \end{cases}$$



Proposition 105 *Let* $\mathcal{U}$ *be a set and let* $A, B \in \mathcal{P}(\mathcal{U})$.

1. $\forall X \in \mathcal{P}(\mathcal{U}).\ A \cup B \subseteq X \iff (A \subseteq X \wedge B \subseteq X)$.
2. $\forall X \in \mathcal{P}(\mathcal{U}).\ X \subseteq A \cap B \iff (X \subseteq A \wedge X \subseteq B)$.

PROOF:

To show that X includes $A \cup B$ equivalently show both that X includes A and X includes B.

PROOF PRINCIPLE:

Corollary 106 *Let* $\mathcal{U}$ *be a set and let* $A, B, C \in \mathcal{P}(\mathcal{U})$.

To show that a set C is the union of two sets A and B, equivalently show

1. $$C = A \cup B$$

   iff

   1 $$[A \subseteq C \wedge B \subseteq C]$$

   $$\wedge$$

   2 $$[\forall X \in \mathcal{P}(\mathcal{U}).\ (A \subseteq X \ \wedge\ B \subseteq X) \implies C \subseteq X]$$

2. $$C = A \cap B$$

   iff

   $$[C \subseteq A \wedge C \subseteq B]$$

   $$\wedge$$

   $$[\forall X \in \mathcal{P}(\mathcal{U}).\ (X \subseteq A \ \wedge\ X \subseteq B) \implies X \subseteq C]$$

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\mathbf{false}\,,\ \mathbf{true}\,\}$
$\emptyset$	**false**
$\mathcal{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{\mathrm{c}}$	$\neg(\cdot)$

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$$\{a, b\} = \{b, a\}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \vee x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{ a \}$, and referred to as a *singleton*.

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Proposition 107 *For all* a, b, c, x, y,

1. $\{a\} = \{x, y\} \iff x = y = a$
2. $\{c, x\} = \{c, y\} \iff x = y$

PROOF:

(1) Assume $\{a\} = \{x, y\}$.

$x \in \{x, y\} = \{a\} \implies x = a$

$y \in$ $\qquad y = a$ $\square$

(2) Assume $\{c, x\} = \{c, y\}$.

$x \in \{c, x\} = \{c, y\} \implies [x = c \vee x = y]$

$y \in \{c, y\} = \{c, x\} \implies [y = c \vee y = x]$

$\implies x = y$ $\square$

Exercise.