Numbers

Objectives

- Get an appreciation for the abstract notion of number system, considering four examples: natural numbers, integers, rationals, and modular integers.

- Prove the correctness of three basic algorithms in the theory of numbers: the division algorithm, Euclid's algorithm, and the Extended Euclid's algorithm.

- Exemplify the use of the mathematical theory surrounding Euclid's Theorem and Fermat's Little Theorem in the context of public-key cryptography.

- To understand and be able to proficiently use the Principle of Mathematical Induction in its various forms.

Natural numbers

In the beginning there were the *natural numbers*

$$\mathbb{N} \; : \quad 0 \,, \quad 1 \,, \quad \ldots \,, \quad n \,, \quad n+1 \,, \quad \ldots$$

generated from *zero* by successive increment; that is, put in ML:

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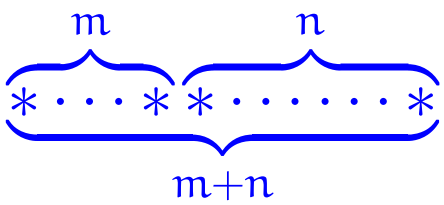
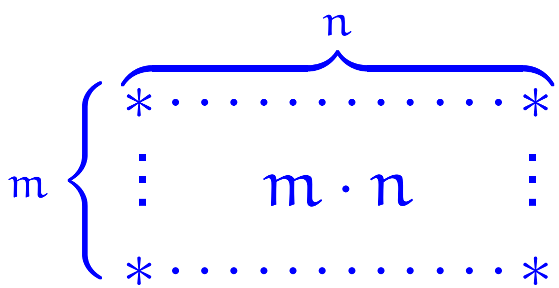
datatype
  N = zero | succ of N
```

The basic operations of this number system are:

- Addition



- Multiplication



neutral element

The *additive structure* $(\mathbb{N}, 0, +)$ of natural numbers with zero and addition satisfies the following:

associativity

- Monoid laws

$$0 + n = n = n + 0 \quad , \quad (l + m) + n = l + (m + n)$$

- Commutativity law

l+m+n

$$m + n = n + m$$

and as such is what in the mathematical jargon is referred to as a *commutative monoid*.

Commutative monoid laws

- Neutral element laws

$$\overbrace{\qquad}^{0}\overbrace{*\cdots\cdots*}^{n} = \overbrace{*\cdots\cdots*}^{n} = \overbrace{*\cdots\cdots*}^{n}\overbrace{\qquad}^{0}$$

- Associativity law

$$\overbrace{*\cdots**\cdots\cdots*}^{\ell+m}\overbrace{*\cdots\cdots*}^{n} = \overbrace{*\cdots*}^{\ell}\overbrace{*\cdots\cdots**\cdots\cdots*}^{m+n}$$

- Commutativity law

$$\overbrace{*\cdots*}^{m}\overbrace{*\cdots\cdots*}^{n} = \overbrace{*\cdots\cdots*}^{n}\overbrace{*\cdots*}^{m}$$

Monoids

Definition 43 *A* monoid *is an algebraic structure with*

- *a* neutral element, *say* e,
- *a* binary operation, *say* $\bullet$,

satisfying

- neutral element laws*:* $e \bullet x = x = x \bullet e$ justifies
- associativity law*:* $(x \bullet y) \bullet z = x \bullet (y \bullet z)$ ⇝ $x \bullet y \bullet z$

Examples: $(\mathbb{N}, 0, +)$ $(\mathbb{N}, 1, \cdot)$ $(\alpha\,list, nil, @)$

int list

Monoids

$[1,2] @ [3,4] \neq [3,4] @ [1,2]$

Definition 43 *A monoid is an algebraic structure with*

- *a neutral element, say* e,
- *a binary operation, say* $\bullet$,

satisfying

- neutral element laws*:* $e \bullet x = x = x \bullet e$
- associativity law*:* $(x \bullet y) \bullet z = x \bullet (y \bullet z)$

A monoid is commutative if:

$(\mathbb{N}, 0, +)$ $(\mathbb{N}, 1, \cdot)$

- commutativity*:* $x \bullet y = y \bullet x$

$(\alpha\ list, nil, @)$ comm $\Leftrightarrow$ $\alpha = unit$

is satisfied.

Also the *multiplicative structure* $(\mathbb{N}, 1, \cdot)$ of natural numbers with one and multiplication is a commutative monoid:

- Monoid laws

$$1 \cdot n = n = n \cdot 1 \quad , \quad (l \cdot m) \cdot n = l \cdot (m \cdot n)$$

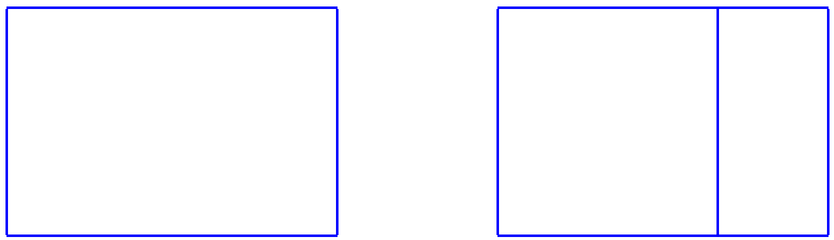
- Commutativity law

$$m \cdot n = n \cdot m$$

The additive and multiplicative structures interact nicely in that they satisfy the

- Distributive laws

$$\begin{aligned} l \cdot 0 &= 0 \\ l \cdot (m + n) &= l \cdot m + l \cdot n \end{aligned}$$



and make the overall structure $(\mathbb{N}, 0, +, 1, \cdot)$ into what in the mathematical jargon is referred to as a *commutative semiring*.

Semirings

Definition 44 *A* semiring *(or* rig*) is an algebraic structure with*

- *a* commutative monoid *structure, say* $(0, \oplus)$, additive structure
- *a* monoid structure, *say* $(1, \otimes)$, multiplicative structure

Semirings

Definition 44 *A semiring (or rig) is an algebraic structure with*

- *a commutative monoid structure, say* $(0, \oplus)$,
- *a monoid structure, say* $(1, \otimes)$,

satifying the distributivity laws:

- $0 \otimes x = 0 = x \otimes 0$
- $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$, $(y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$

Examples: $(\mathbb{N}, 0, +, 1, \cdot)$

Semirings

Definition 44 *A* semiring *(or* rig*) is an algebraic structure with*

- *a* commutative monoid *structure, say* $(0, \oplus)$,
- *a* monoid structure, *say* $(1, \otimes)$,

satifying the distributivity laws*:*

- $0 \otimes x = 0 = x \otimes 0$
- $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$, $(y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$

A semiring is commutative *whenever* $\otimes$ *is.*

Cancellation

The additive and multiplicative structures of natural numbers further satisfy the following laws.

- Additive cancellation

  For all natural numbers k, m, n,

  $$k + m = k + n \implies m = n \quad .$$

- Multiplicative cancellation

  For all natural numbers k, m, n,

  if $k \neq 0$ then $k \cdot m = k \cdot n \implies m = n \quad .$

Definition 45 *A binary operation* $\bullet$ *allows* cancellation *by an element* c

- on the left*: if* $c \bullet x = c \bullet y$ *implies* $x = y$
- on the right*: if* $x \bullet c = y \bullet c$ *implies* $x = y$

Example: The append operation on lists allows cancellation by any list on both the left and the right.

$$\ell \,@\, \ell_1 = \ell \,@\, \ell_2 \Rightarrow \ell_1 = \ell_2$$

Inverses

Definition 46 *For a monoid with a neutral element* e *and a binary operation* $\bullet$, *and element* x *is said to admit an*

- inverse on the left *if there exists an element* ℓ *such that* $\ell \bullet x = e$
- inverse on the right *if there exists an element* r *such that* $x \bullet r = e$
- inverse *if it admits both left and right inverses*

Examples: $(\mathbb{N}, 0, +)$ ✗ ⇝ $(\mathbb{Z}, 0, +)$ ✓

Inverses

Definition 46 *For a monoid with a neutral element* e *and a binary operation* $\bullet$, *and element* x *is said to admit an*

- inverse on the left *if there exists an element* ℓ *such that* $\ell \bullet x = e$
- inverse on the right *if there exists an element* r *such that* $x \bullet r = e$
- inverse *if it admits both left and right inverses* — Typically x^{-1}

Proposition 47 *For a monoid* $(e, \bullet)$ *if an element admits an inverse then its left and right inverses are equal.*

PROOF: Let x have left inverse ℓ and right inverse r.

$r = e \cdot r = (\ell \cdot x) \cdot r = \ell \cdot x \cdot r = \ell \cdot (x \cdot r) = \ell \cdot e = \ell$ ⊠

Groups

Definition 49 *A* group *is a monoid in which every element has an inverse.*

An Abelian group *is a group for which the monoid is commutative.*

<u>Examples</u>: $(\mathbb{Z}, 0, +, -)$ integers (modular)

Inverses

Definition 50

1. *A number* x *is said to admit an* additive inverse *whenever there exists a number* y *such that* $x + y = 0$.
2. *A number* x *is said to admit a* multiplicative inverse *whenever there exists a number* y *such that* $x \cdot y = 1$.

Extending the system of natural numbers to: (i) admit all additive inverses and then (ii) also admit all multiplicative inverses for non-zero numbers yields two very interesting results:

(i) the *integers*

$$\mathbb{Z} \ : \quad \ldots -n\,,\ \ldots\,,\ -1\,,\ 0\,,\ 1\,,\ \ldots\,,\ n\,,\ \ldots$$

which then form what in the mathematical jargon is referred to as a *commutative ring*, and

(ii) the *rationals* $\mathbb{Q}$ which then form what in the mathematical jargon is referred to as a *field*.

Rings

Definition 51 *A* ring *is a semiring* $(0, \oplus, 1, \otimes)$ *in which the commutative monoid* $(0, \oplus)$ *is a group.*

A ring is commutative *if so is the monoid* $(1, \otimes)$.

Fields

Definition 52 *A* field *is a commutative ring in which every element besides* 0 *has a reciprocal (that is, and inverse with respect to* $\otimes$*).*

$q_1 \cdot n + r_1 = m = q_2 \cdot n + r_2 \Longrightarrow q_1 \cdot n = q_2 \cdot n \Longrightarrow q_1 = q_2$. ☒

(cancellation) (cancellation)

The division theorem and algorithm

Theorem 53 (Division Theorem) *For every natural number* m *and positive natural number* n*, there exists a unique pair of integers* q *and* r *such that* $q \geq 0$, $0 \leq r < n$, *and* $m = q \cdot n + r$.

Uniqueness:

$q_1 \geq 0,\ 0 \leq r_1 < n,\ m = q_1 \cdot n + r_1$

and

$q_2 \geq 0,\ 0 \leq r_2 < n,\ m = q_2 \cdot n + r_2$ (*)

$\Longrightarrow q_1 = q_2$ and $r_1 = r_2$

Assume (*):

$m \equiv r_1 \pmod n$

$m \equiv r_2 \pmod n$

$\Rightarrow r_1 \equiv r_2 \pmod n \Rightarrow r_1 = r_2$ (previously shown)