Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$*, the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{\perp\}$, we have

$$\boxed{[\![M]\!] \stackrel{\text{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

$$\textit{if} \quad [\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\textit{then} \quad \big[\!\big[\Gamma' \vdash \mathcal{C}[M]\big]\!\big] = \big[\!\big[\Gamma' \vdash \mathcal{C}[M]\big]\!\big] : [\![\Gamma']\!] \to [\![\tau']\!]$$

For instance,

$$C[\,] = [\,]\, N$$

$$[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!]$$

$$\Rightarrow [\![\Gamma \vdash M N]\!] \stackrel{?}{=} [\![\Gamma \vdash M' N]\!]$$

$$[\![\Gamma \vdash M N]\!](\rho) \stackrel{?}{=} [\![\Gamma \vdash M' N]\!](\rho) \quad \forall \rho$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$[\![\Gamma \vdash M]\!](\rho)\left([\![\Gamma \vdash N]\!](\rho)\right) \qquad [\![\Gamma \vdash M']\!](\rho)\left([\![\Gamma \vdash N]\!]\rho\right)$$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

$$\text{if } M \Downarrow_\tau V \text{ then } [\![M]\!] = [\![V]\!] \in [\![\tau]\!] \ .$$

For instance,

$$\frac{M \Downarrow \underline{succ}(V)}{\underline{pred}(M) \Downarrow V}$$

By induction $[\![M]\!] = [\![succ(V)]\!] = [\![V]\!] + 1$

$\underline{RTP}$: $[\![pred(M)]\!] \overset{?}{=} [\![V]\!]$

$$\parallel$$

$$p\,[\![M]\!] = \begin{cases} \perp & [\![M]\!] = 0, \perp \\ [\![M]\!] - 1 & \text{otw} \end{cases} = [\![V]\!] + 1 - 1 = [\![V]\!]$$

For instance,

$$\frac{M \Downarrow \mathrm{fn}\, x. M' \qquad M'[N/x] \Downarrow V}{M N \Downarrow V}$$

By induction,

$$[\![M]\!] = [\![\mathrm{fn}\, x. M']\!] = \lambda d.\, [\![x \vdash M']\!][x \mapsto d]$$

$$[\![M'[N/x]]\!] = [\![V]\!]$$

? ~ Lemma.

RTP:

$$[\![M N]\!] \stackrel{?}{=} [\![V]\!]$$

$$\|$$

$$[\![M]\!]([\![N]\!]) = [\![x \vdash M']\!][x \mapsto [\![N]\!]]$$

$$\frac{\Gamma, x:\sigma \vdash M:\tau \qquad \Gamma \vdash N:\sigma}{}$$

$$[\![\Gamma, x:\sigma \vdash M:\tau]\!] : \underbrace{[\![\Gamma, x:\sigma]\!]}_{\prod_{(x_i:\tau_i)\in(\Gamma, x:\sigma)} [\![\tau_i]\!]} \to [\![\tau]\!]$$

$$\frac{[\![\Gamma \vdash N:\sigma]\!] : [\![\Gamma]\!] \to [\![\sigma]\!]}{[\![\Gamma \vdash M[N/x]:\tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]}$$

Lemma

$$\lambda \rho \in [\![\Gamma]\!].\ [\![\Gamma, x:\sigma \vdash M:\tau]\!]\Big(\rho[x \mapsto \underbrace{[\![\Gamma \vdash N:\sigma]\!](\rho)}_{\text{in } [\![\sigma]\!]}]\Big)$$

Lemma: $\Gamma \vdash M : \tau \qquad [\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$

$x \notin \underline{\mathrm{dom}}(\Gamma)$

$\Gamma, x:\sigma \vdash M : \tau$

$$\begin{tikzcd}
{[\![\Gamma, x:\sigma]\!]} \arrow[rr, "{[\![\Gamma, x:\sigma \vdash M]\!]}"] \arrow[dr, "\text{proj.}"'] & \underline{\text{Lemma}} & {[\![\tau]\!]} \\
& {[\![\Gamma]\!]} \arrow[ur, "{[\![\Gamma \vdash M]\!]}"'] &
\end{tikzcd}$$

$\Gamma = (x_1 : \tau_1 \ldots x_n : \tau_n)$

$[\![\Gamma, x:\sigma \vdash M]\!](d_1, d_2, \ldots, d_n, d) = [\![\Gamma \vdash M]\!](d_1, \ldots, d_n)$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$\begin{aligned}&[\![\Gamma \vdash M'[M/x]]\!](\rho)\\ &\qquad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!](\rho)\big]\big)\end{aligned}$$

for all $\rho \in [\![\Gamma]\!]$.

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$

Generalities of denotational semantics

- Domains.
  - interpretations for nat, bool, for functions and products.
    $(D \to E)$ $\qquad$ $(D_1 \times \cdots \times D_n)$

- Terms
  - interpretations for successor, predecessor, ...
    fix: $(D \to D) \to D$

    pairing: $\dfrac{D \to E \qquad D \to F}{D \to E \times F}$

    projections: $D_1 \times \cdots \times D_n \longrightarrow D_i$

currying:

$$\frac{D \times E \to F}{D \to (E \to F)}$$

evaluation: $(D \to E) \times D \longrightarrow E$

Examples:

- Scott domains (cpos).
- Stable domains and functions

Topic 7

Relating Denotational and Operational Semantics

Adequacy

For any closed PCF terms M and V of *ground* type $\gamma \in \{nat, bool\}$ with V a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V\ .$$

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NB. Adequacy does not hold at function types:

$$[\![\mathbf{fn}\, x : \tau .\, (\mathbf{fn}\, y : \tau .\, y)\, x]\!] \quad = \quad [\![\mathbf{fn}\, x : \tau .\, x]\!] \quad : [\![\tau]\!] \to [\![\tau]\!]$$

but

$$\mathbf{fn}\, x : \tau .\, (\mathbf{fn}\, y : \tau .\, y)\, x \not\Downarrow_{\tau \to \tau} \mathbf{fn}\, x : \tau .\, x$$

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$.

$$[\![M_1\, M_2]\!] = [\![V]\!] \in [\![\gamma]\!] \overset{?}{\Longrightarrow} M_1\, M_2 \Downarrow_\gamma V$$

with

$M_1 : \sigma \to \gamma$

$M_2 : \sigma$

By ind$\overset{?}{:}$ $[\![M_1]\!] \in ([\![\sigma]\!] \to [\![\gamma]\!])$

$[\![M_2]\!] \in [\![\sigma]\!]$

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2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider M to be $M_1\, M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

   $$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.