Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.

Check :

① $g \circ (-)$ is cont. $\forall g$

② $(-) \circ f$ is cont. $\forall f$.

① (i) monotonicity

$$f \sqsubseteq f' \overset{?}{\Rightarrow} g \circ f \sqsubseteq g \circ f'$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\forall x \; f(x) \sqsubseteq f'(x) \xrightarrow[\text{by monotonicity of } g]{} \forall y \; g(f y) \sqsubseteq g(f' y)$$

(ii) $f_0 \sqsubseteq f_1 \sqsubseteq \dots \sqsubseteq f_n \sqsubseteq \dots$

$$g \circ \left(\bigsqcup_n f_n\right) = \bigsqcup_n (g \circ f_n)$$

$$g \circ \left(\bigsqcup_n f_n\right) \overset{?}{=} \bigsqcup_n (g \circ f_n)$$

$$\Leftrightarrow \forall x.\quad g\left(\left(\bigsqcup_n f_n\right)(x)\right) \overset{?}{=} \left(\bigsqcup_n (g \circ f_n)\right)(x)$$

$$\begin{array}{ccc} g\left(\left(\bigsqcup_n f_n\right)(x)\right) & & \left(\bigsqcup_n (g \circ f_n)\right)(x) \\ \| & & \| \\ g\left(\bigsqcup_n (f_n(x))\right) & & \bigsqcup_n (g \circ f_n)(x) \\ \| & & \\ \bigsqcup_n g(f_n(x)) & = & \end{array}$$

② Exercise.

Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.

$$\mathit{fix}(f) = \bigsqcup_n f^n(\bot)$$

① fix monotone

$$f \sqsubseteq g \Rightarrow \mathit{fix}(f) \overset{?}{\sqsubseteq} \mathit{fix}(g)$$

$$\mathit{fix}(f) = \bigsqcup_n f^n(\bot) \qquad \mathit{fix}(g) = \bigsqcup_n g^n(\bot)$$

$$f \sqsubseteq g \iff (\circledast)\ f(x) \sqsubseteq g(x)$$

$$\Uparrow$$

$$f^n(\bot) \sqsubseteq g^n(\bot) \sim \text{by induction}$$

$$\Downarrow$$

$n=0$ $\quad \bot \sqsubseteq \bot$

$n=1$ $\quad f(\bot) \sqsubseteq g(\bot)$

$n=2$ $\quad f(f\,\bot) \underset{\text{mon } f}{\sqsubseteq} f(g\,\bot) \underset{\circledast}{\sqsubseteq} g(g\,\bot)$

② $f_0 \sqsubseteq f_1 \sqsubseteq \dots \sqsubseteq f_n \sqsubseteq \dots$

$$\mathrm{fix}\left(\bigsqcup_n f_n\right) \overset{?}{=} \bigsqcup_n \mathrm{fix}(f_n)$$

Since fix is monotone:

$$\bigsqcup_n \underline{\mathrm{fix}}(f_n) \sqsubseteq \underline{\mathrm{fix}}\left(\bigsqcup_n f_n\right)$$

RTP

$$\frac{\left(\bigsqcup_n f_n\right)\left(\bigsqcup_n \underline{\mathrm{fix}}\, f_n\right) \sqsubseteq \bigsqcup_n \underline{\mathrm{fix}}(f_n)}{\underline{\mathrm{fix}}\left(\bigsqcup_n f_n\right) \sqsubseteq \bigsqcup_n \underline{\mathrm{fix}}(f_n)}$$

h monotone

$$h\left(\bigsqcup_n d_n\right)$$
$$\sqcup\!\!\mid$$
$$\bigsqcup_n h(d_n)$$

show
$$h(d_n) \sqsubseteq h\left(\bigsqcup_n d_n\right)$$
show
$$d_n \sqsubseteq \bigsqcup_n d_n$$

$$\frac{e_n \sqsubseteq e}{\bigsqcup_n e_n \sqsubseteq e}$$

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_k \underline{fix}\, f_k\right) \overset{?}{\sqsubseteq} \bigsqcup_n \underline{fix}(f_n)$$

$$\shortparallel$$

$$\bigsqcup_n f_n\left(\bigsqcup_k \underline{fix}(f_k)\right)$$

$$\shortparallel$$

$$\bigsqcup_n \bigsqcup_k f_n\left(\underline{fix}(f_k)\right)$$

$$\shortparallel$$

$$\bigsqcup_n f_n\left(\underline{fix}(f_n)\right)$$

$$\sim \quad f(\underline{fix}\, f) \sqsubseteq \underline{fix}\, f$$

Topic 4

Scott Induction

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$\mathit{fix}(f) \in S \ ,$$

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

$$\frac{d \in S \Rightarrow f(d) \in S}{\underline{\mathit{fix}}(f) \in S} \ (S \text{ admissible})$$

$$\left(\forall d \quad d \in S \Rightarrow f(d) \in S \right) \Rightarrow \underline{fix}(f) \in S$$

$\boxed{?}$ What kind of S supports the proof principle?

$$\underline{fix}(f) = \bigsqcup_n f^n(\bot)$$

$\circledast$ $\bot \in S \Rightarrow f(\bot) \in S \Rightarrow f\,f(\bot) \in S \Rightarrow \cdots$

$\Rightarrow \forall n \; f^n(\bot) \in S$

$\circledast$ $(d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots)$ in S

$\Rightarrow \left(\bigsqcup_n d_n\right)$ in S

$\Rightarrow \underline{fix}(f) \in S$

$$\Omega = (0 < 1 < 2 < \cdots < n < \cdots < \omega)$$

$$S' = \{n \mid n \in \mathbb{N}\} \subseteq \Omega$$

$$\Omega \times \Omega$$

$$S = \{(n,n) \mid n \in \mathbb{N}\} \subseteq \Omega \times \Omega$$

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\vee & & & & & & \\
(2,0) & < & \cdots & & & & \\
\vee & & & & & & \\
(1,0) & < & (1,1) & < & \cdots & & \\
\vee & & \vee & & & & \\
(0,0) & < & (0,1) & < & (0,2) & < & \cdots
\end{array}$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n \geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called *chain-closed* (resp. *admissible*) iff $\{d \in D \mid \Phi(d)\}$ is a *chain-closed* (resp. *admissible*) subset of D.

Building chain-closed subsets (I)

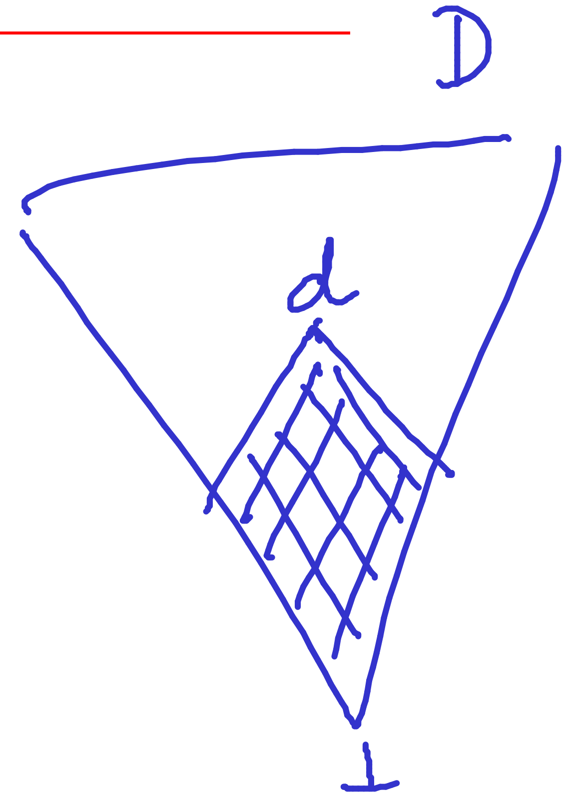
Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

of D is chain-closed.



Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

of D is chain-closed.

$(x_0, y_0) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots$ in S

s.t. $x_0 \sqsubseteq y_0, \cdots, x_n \sqsubseteq y_n$

- The subsets

$$S = \{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

and

$$\{(x, y) \in D \times D \mid x = y\}$$

of $D \times D$ are chain-closed.

$\bigsqcup_n (x_n, y_n)$

$= (\bigsqcup_n x_n, \bigsqcup_n y_n)$ in S

$\Leftrightarrow \bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$ ✓

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

$$\Updownarrow$$

$$\mathit{fix}(f) \in \downarrow(d)$$

$$x \sqsubseteq d \overset{?}{\Longrightarrow} f(x) \sqsubseteq d$$

$$\Downarrow$$

$$f(x) \sqsubseteq f(d) \sqsubseteq d$$

$$\forall x \quad \frac{x \in \downarrow(d) \Rightarrow f(x) \in \downarrow(d)}{\mathit{fix}(f) \in \downarrow(d)} \; (\downarrow(d) \text{ adm.})$$

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in {\downarrow}(d) &\implies x \sqsubseteq d \\ &\implies f(x) \sqsubseteq f(d) \\ &\implies f(x) \sqsubseteq d \\ &\implies f(x) \in {\downarrow}(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in {\downarrow}(d) \ .$$

Building chain-closed subsets (II)

Inverse image:

Let $f : D \to E$ be a continuous function.

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S \;=\; \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of D.

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \; .$$

Instinct: use the admissible subset $\downarrow(\mathit{fix}(g))$.

$\bot \in \{x \mid f(x) \sqsubseteq g(x)\}$ ~ chain closed (Exercise).

$$\frac{f(\mathit{fix}(g)) \sqsubseteq \mathit{fix}(g) = g(\mathit{fix}(g))}{\mathit{fix}(f) \sqsubseteq \mathit{fix}(g)}$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\perp) \sqsubseteq g(\perp) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.

- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.