Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.

(1) $f(\underline{\mathrm{fix}}(f)) \overset{?}{\sqsubseteq} \underline{\mathrm{fix}}(f) = \bigsqcup_n \underbrace{f^n(\bot)}_{\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots}$

$$\begin{aligned}
&f(\underline{\mathrm{fix}}(f)) \\
&\quad = f\Big(\bigsqcup_n f^n(\bot)\Big) \\
&\quad = \bigsqcup_n f(f^n(\bot)) = \bigsqcup_n \underbrace{f^{n+1}(\bot)}_{f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \cdots}
\end{aligned}$$

$\bigsqcup_n f^n(\bot) \;//\; \bigsqcup_n f^{n+1}(\bot)$

(2) $\forall d.\ \underline{f(d) \sqsubseteq d}$?

RTP $\underline{\mathrm{fix}}(f) \sqsubseteq d$

$$\underline{\mathrm{fix}}(f) = \bigsqcup_n f^n(\bot)$$

$$\begin{aligned}
&\bot \sqsubseteq d \\
&f(\bot) \sqsubseteq f(d) \sqsubseteq d \\
&f^2(\bot) \sqsubseteq f(d) \sqsubseteq d \\
&\cdots
\end{aligned}$$

$$\forall n.\ f^n(\bot) \sqsubseteq d$$

Topic 3

Constructions on Domains

datatypes

✓ $\alpha * \beta$

$cons_1$ of α_1 | ... | $cons_n$ of α_n

✓ $\alpha \to \beta$

recursive types

Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

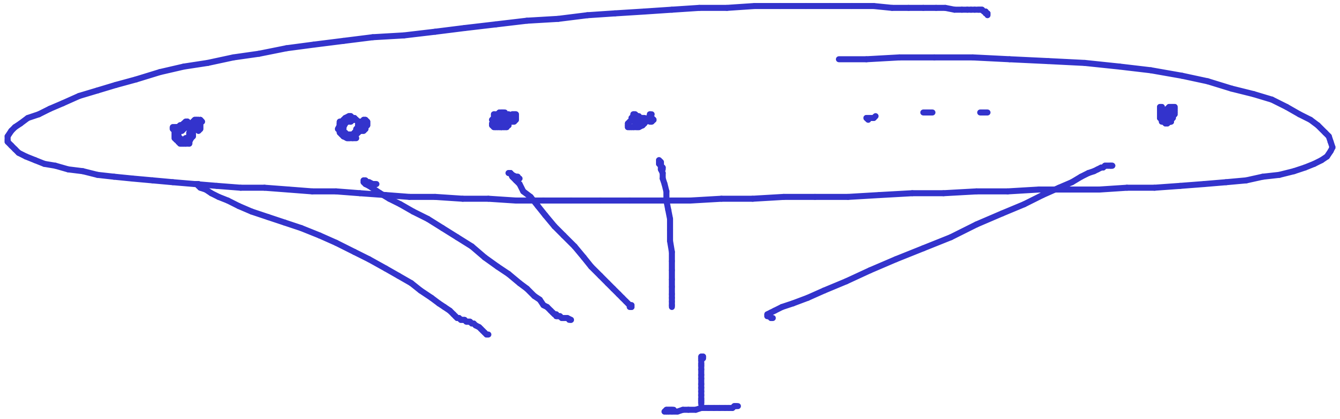
makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

X



<u>examples</u>: $\{true, false\}_\perp$

$\mathbb{N}_\perp$

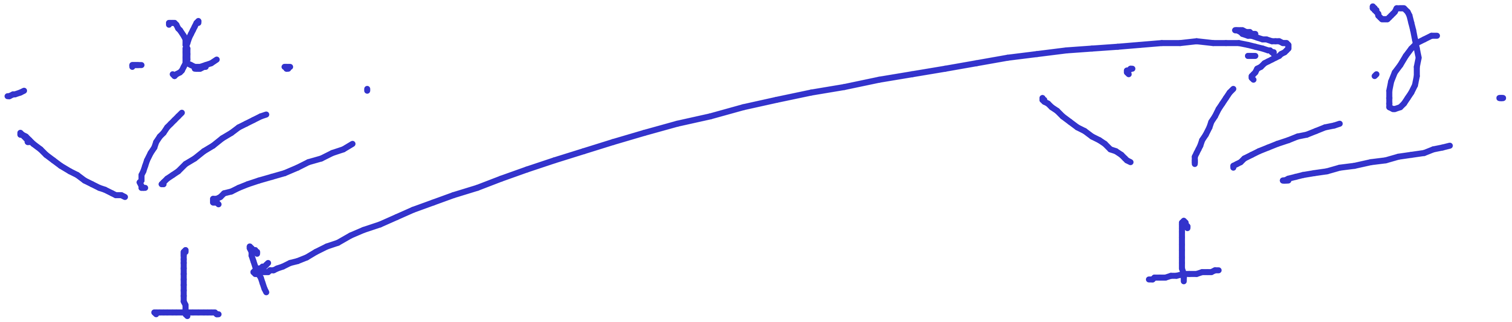
X, Y sets

$$X_\perp \xrightarrow[\text{cont.}]{f} Y_\perp$$

$\Updownarrow$

monotone & preserve lubs

Suppose (1) $f(\perp) = y \in Y \Rightarrow \forall x \in X \;\; \underset{\substack{\| \\ y}}{f(\perp)} \sqsubseteq f(x) \Rightarrow f(x) = y$



(2) $f(\perp) = \perp$

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \ \& \ d_2 \sqsubseteq_2 d_2' \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

$(D_1, \sqsubseteq_1)$ $(D_2, \sqsubseteq_2)$

$(D_1 \times D_2, \sqsubseteq)$ ~ partial order
- reflexivity
- antisymmetry
- transitivity

~ complete

~ least element.

$d_1 \sqsubseteq_1 d_1'$
$d_2 \sqsubseteq_2 d_2'$ $\Longleftrightarrow$ $(d_1, d_2) \sqsubseteq (d_1', d_2')$

$d_1' \sqsubseteq_1 d_1''$
$d_2' \sqsubseteq_2 d_2''$ $\Longleftrightarrow$ $(d_1', d_2') \sqsubseteq (d_1'', d_2'')$

$\overset{?}{\Longrightarrow}$ $(d_1, d_2) \sqsubseteq (d_1'', d_2'')$

$d_1 \sqsubseteq d_1''$

$d_2 \sqsubseteq d_2''$

$$(d_0, e_0) \sqsubseteq (d_1, e_1) \sqsubseteq (d_2, e_2) \sqsubseteq \cdots \sqsubseteq (d_n, e_n) \sqsubseteq \cdots$$

in $D_1 \times D_2$

$\Updownarrow$ $\quad$ $\Updownarrow$

$d_0 \sqsubseteq d_1$ $\quad$ $d_1 \sqsubseteq d_2$ $\quad \cdots$

$e_0 \sqsubseteq e_1$ $\quad$ $e_1 \sqsubseteq e_2$

So $\quad d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ in D_1

and $\quad e_0 \sqsubseteq e_1 \sqsubseteq \cdots \sqsubseteq e_n \sqsubseteq \cdots$ in D_2

Giving $\bigsqcup_n d_n$ in D_1 and $\bigsqcup_n e_n$ in D_2

and so $\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right)$ in $D_1 \times D_2$

<u>Claim</u>: $(d_0, e_0) \sqsubseteq (d_1, e_1) \sqsubseteq \cdots \sqsubseteq (d_n, e_n) \sqsubseteq \cdots$

has lub $\left(\bigsqcup_n d_n, \bigsqcup_n e_n\right)$

In other words

$$\bigsqcup_n (d_n, e_n) = \left(\bigsqcup_n d_n, \bigsqcup_n e_n\right).$$

(1) $(d_k, e_k) \sqsubseteq \left(\bigsqcup_n d_n, \bigsqcup_n e_n\right) \quad \forall k$

$\Longleftrightarrow \quad d_k \sqsubseteq_1 \bigsqcup_n d_n$ and $e_k \sqsubseteq_2 \bigsqcup_n e_n$

(2) $\forall k.\, (d_k, e_k) \sqsubseteq (x, y) \overset{?}{\Longrightarrow} \left(\bigsqcup_n d_n, \bigsqcup_n e_n\right) \sqsubseteq (x, y)$

$\Downarrow$

$d_k \sqsubseteq x \wedge e_k \sqsubseteq y \Longrightarrow \bigsqcup_n d_n \sqsubseteq x \wedge \bigsqcup_n e_n \sqsubseteq y$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

Continuous functions of two arguments

Proposition. *Let* D, E, F *be cpo's. A function* $f : (D \times E) \to F$ *is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d, e) \sqsubseteq f(d', e)$$

$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d, e) \sqsubseteq f(d, e').$$

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

$$f(\bigsqcup_{m \geq 0} d_m \,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$

Exercise

$$f(d \,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

$f : D \times E \to F$ monotone

$(d,e) \sqsubseteq (d',e') \Rightarrow f(d,e) \sqsubseteq f(d',e')$ ③

Then

① $d \sqsubseteq d' \Rightarrow f(d,e) \sqsubseteq f(d',e) \quad \forall e.$

② $e \sqsubseteq e' \Rightarrow f(d',e) \sqsubseteq f(d',e') \quad \forall d'$

Also

① $\wedge$ ② $\Rightarrow$ ③

Because if $(d,e) \sqsubseteq (d',e')$ Then $d \sqsubseteq d'$ and $e \sqsubseteq e'$

and $f(d,e) \sqsubseteq f(d',e) \wedge f(d',e) \sqsubseteq f(d',e')$

So we are done.

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (f \text{ continuous})$$

$$f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_m f(x_m, \bigsqcup_n y_n)$$

$$= \bigsqcup_m \bigsqcup_n f(x_m, y_n)$$

$$= \bigsqcup_k f(x_k, y_k).$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

$f \sqsubseteq f$

$f \sqsubseteq g \wedge g \sqsubseteq h \stackrel{?}{\Rightarrow} f \sqsubseteq h$

$f(d) \sqsubseteq_E g(d) \quad g(d) \sqsubseteq_E h(d) \Rightarrow f(d) \sqsubseteq h(d) \ \forall d$

$f \sqsubseteq g \wedge g \sqsubseteq f \Rightarrow f = g.$

$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$ in $(D \to E)$

$\Rightarrow$ $f_0(d) \sqsubseteq f_1(d) \sqsubseteq \cdots \sqsubseteq f_n(d) \sqsubseteq \cdots$ in E

Define $f : D \to E$

by $f(d) \stackrel{\text{def}}{=} \bigsqcup_n \underbrace{(f_n(d))}_{\text{a chain in } E}$

<u>Claim</u>: f is continuous.

(1) f monotone ———— $\dfrac{d \sqsubseteq d'}{f(d) \sqsubseteq f(d')}$?

$f(d) = \bigsqcup_n f_n(d)$, $\quad f(d') = \bigsqcup_n f_n(d')$

(2) f preserves lubs.

f_n monotone

$d \sqsubseteq d' \Rightarrow f_n(d) \sqsubseteq f_n(d')$

$$f\Big(\bigsqcup_k d_k\Big) \overset{?}{=} \bigsqcup_k f(d_k)$$

$$\bigsqcup_n f_n\Big(\bigsqcup_k d_k\Big) \qquad\qquad \bigsqcup_k \bigsqcup_n f_n(d_k)$$

$\parallel$ ~ f_n cont.

$$\bigsqcup_n \bigsqcup_k f_n(d_k) = \bigsqcup_k \bigsqcup_n f_n(d_k)$$

Claim: f is the lub of $f_0 \sqsubseteq f_1 \sqsubseteq \dots \sqsubseteq f_n \sqsubseteq \dots$

(1) $f_n \sqsubseteq f \iff f_n(d) \sqsubseteq f(d) = \bigsqcup_n f_n(d)$ ✓

(2) $f_n \sqsubseteq g \overset{?}{\Longrightarrow} f \sqsubseteq g$

$$f_n \sqsubseteq g \Rightarrow f_n(d) \sqsubseteq g(d) \quad \forall d$$

$$\Rightarrow f(d) = \bigsqcup_n f_n(d) \sqsubseteq g(d) \quad \forall d$$

$$\Rightarrow f \sqsubseteq g .$$

$\boxtimes$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n \geq 0} f_n \;=\; \lambda d \in D.\, \bigsqcup_{n \geq 0} f_n(d)\;.$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D \to E}(d) = \perp_E$, all $d \in D$.