Pre-fixed points

Let D be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{fix(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(fix(f)) \sqsubseteq fix(f) \tag{lfp1}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \ \Rightarrow\ fix(f) \sqsubseteq d. \tag{lfp2}$$

Proof principle

1.
$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x} \text{(lfp2)}$$

$(\mathbb{N}, \leq)$ $\quad$ $succ : \mathbb{N} \to \mathbb{N} : n \mapsto n+1$

It is monotone but does not have a least pre-fixed point.

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)}$$

$$\dfrac{\dfrac{}{f(\underline{\mathrm{fix}}(f)) \sqsubseteq \underline{\mathrm{fix}}(f)}\,\mathrm{lfp1} \qquad (f\ \text{monotone})\ \dfrac{\dfrac{\dfrac{}{f(\underline{\mathrm{fix}}\ f) \sqsubseteq \underline{\mathrm{fix}}(f)}\,\mathrm{lfp}\ 1}{f(f(\underline{\mathrm{fix}}\ f)) \sqsubseteq f(\underline{\mathrm{fix}}\ f)}}{\underline{\mathrm{fix}}(f) \sqsubseteq f\,(\underline{\mathrm{fix}}(f))}\,\mathrm{lfp2}}{f(\mathrm{fix}(f)) = \mathrm{fix}(f)}$$

D

$\bigsqcup_n d_n \sqsubseteq x^{\forall}$

$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

Thesis*

All domains of computation are
complete partial orders with a least element.

All computable functions are
continuous.

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \qquad \text{(lub1)}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \qquad \text{(lub2)}$$

A domain is a cpo that possesses a least element, $\perp$:

$$\forall d \in D \,.\, \perp \sqsubseteq d.$$

$$\frac{}{\bot \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$f \sqsubseteq g \quad \text{iff} \quad dom(f) \subseteq dom(g) \text{ and } \forall x \in dom(f).\ f(x) = g(x)$$

$$\text{iff} \quad graph(f) \subseteq graph(g)$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

$graph(f) = \bigcup_{n \geq 0} graph(f_n)$

$\bigsqcup_{n \geq 0} f_n$ is the partial function f s.t.

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Least element $\perp$ is the totally undefined partial function.

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

   for all $N \in \mathbb{N}$.

$\boxed{?}$ $d_k \sqsubseteq d_{N+k}$?

$$d_k \sqsubseteq d_{k+1} \sqsubseteq d_{k+2} \sqsubseteq \cdots \sqsubseteq d_{k+N}$$

$$\dfrac{\dfrac{\dfrac{}{d_n \sqsubseteq d_{N+n}} \qquad \dfrac{}{d_{N+n} \sqsubseteq \bigsqcup_k d_{N+k}}}{d_n \sqsubseteq \bigsqcup_k d_{N+k}}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_k d_{N+k}} \qquad \dfrac{\dfrac{}{d_{N+k} \sqsubseteq \bigsqcup_n d_n}}{\bigsqcup_k d_{N+k} \sqsubseteq \bigsqcup_n d_n}$$

$$\dfrac{}{\bigsqcup_n d_n = \bigsqcup_k d_{N+k}}$$

$$e_0 \sqsubseteq e_1 \sqsubseteq \cdots \sqsubseteq e_n \sqsubseteq \cdots \sqsubseteq \bigsqcup_n e_n$$
$$\sqcup\!\text{I} \quad \sqcup\!\text{I} \quad \cdots \quad \sqcup\!\text{I} \quad \cdots \Longrightarrow \quad \sqcup\!\text{I}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $\sqsubseteq \bigsqcup_n d_n$
   $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$(\underbrace{\underline{Ch}(D)}_{\text{The set of chains in } D}, \underbrace{\sqsubseteq}_{(d_n)_{n\in\mathbb{N}} \sqsubseteq (e_n)_{n\in\mathbb{N}} \iff \forall n\in\mathbb{N}\ d_n \sqsubseteq_D e_n})$$

$$\underbrace{\underline{Ch}(D) \xrightarrow{\sqcup} D}_{\text{is monotone}} : (d_n)_{n\in\mathbb{N}} \mapsto \bigsqcup_n d_n$$

$$\dfrac{\dfrac{\overline{d_n \sqsubseteq e_n} \qquad \overline{e_n \sqsubseteq \bigsqcup_k e_k}}{d_n \sqsubseteq \bigsqcup_k e_k}}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})$$

$$
\begin{tikzcd}
\bigsqcup_m d_{m,0} \arrow[r, phantom, "\sqsubseteq"] & \bigsqcup_m d_{m,1} \arrow[r, phantom, "\sqsubseteq"] & \cdots \arrow[rr, phantom, "\sqsubseteq"] & & \bigsqcup_n\left(\bigsqcup_m d_{m,n}\right) & \\
\vdots \arrow[u, phantom, "\sqsubseteq" sloped] & \vdots \arrow[u, phantom, "\sqsubseteq" sloped] & \vdots \arrow[u, phantom, "\sqsubseteq" sloped] & & & \\
d_{m,0} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & d_{m,1} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & d_{m,2} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & \cdots \arrow[r, phantom, "\sqsubseteq"] & d_{m,n} \arrow[r, phantom, "\sqsubseteq"] & \cdots \\
\vdots \arrow[u, phantom, "\sqsubseteq" sloped] & \vdots \arrow[u, phantom, "\sqsubseteq" sloped] & \vdots \arrow[u, phantom, "\sqsubseteq" sloped] & & \vdots \arrow[u, phantom, "\sqsubseteq" sloped] & \vdots \arrow[u, phantom, "\sqsubseteq" sloped] \\
d_{1,0} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & d_{1,1} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & d_{1,2} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & \cdots & d_{1,n} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & \cdots \quad \bigsqcup_n d_{1,n} \arrow[u, phantom, "\sqsubseteq" sloped] \\
d_{0,0} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & d_{0,1} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & d_{0,2} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & \cdots \arrow[r, phantom, "\sqsubseteq"] & d_{0,n} \arrow[u, phantom, "\sqsubseteq" sloped] \arrow[r, phantom, "\sqsubseteq"] & \cdots \quad \bigsqcup_n d_{0,n} \arrow[u, phantom, "\sqsubseteq" sloped]
\end{tikzcd}
$$

$$d_{0,0} \sqsubseteq d_{1,1} \sqsubseteq d_{2,2} \sqsubseteq \cdots \sqsubseteq d_{k,k} \sqsubseteq \cdots \sqsubseteq \bigsqcup_k d_{k,k}$$

$$\bigsqcup_n\left(\bigsqcup_m d_{m,n}\right) = \bigsqcup_k d_{k,k} = \bigsqcup_m\left(\bigsqcup_n d_{m,n}\right)$$

$$\bigsqcup_k d_{k,k}$$

$$\sqsupseteq$$

$$\dfrac{\dfrac{\dfrac{d_{m,n} \sqsubseteq d_{\max(m,n),\max(m,n)}}{d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}}{\bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}}{\bigsqcup_m \left(\bigsqcup_n d_{m,n} \right) \sqsubseteq \bigsqcup_k d_{k,k}} \qquad \dfrac{\dfrac{d_{k,k} \sqsubseteq \bigsqcup_n d_{k,n} \sqsubseteq \bigsqcup_m \left(\bigsqcup_n d_{m,n} \right)}{d_{k,k} \sqsubseteq \bigsqcup_m \left(\bigsqcup_n d_{m,n} \right)}}{\bigsqcup_k d_{k,k} \sqsubseteq \bigsqcup_m \left(\bigsqcup_n d_{m,n} \right)}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$\bigsqcup_m \left(\bigsqcup_n d_{m,n} \right) = \bigsqcup_k d_{k,k}$$

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \ \& \ n \leq n' \ \Rightarrow \ d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n\geq 0} d_{0,n} \ \sqsubseteq \ \bigsqcup_{n\geq 0} d_{1,n} \ \sqsubseteq \ \bigsqcup_{n\geq 0} d_{2,n} \ \sqsubseteq \ \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \ \sqsubseteq \ \bigsqcup_{m\geq 0} d_{m,1} \ \sqsubseteq \ \bigsqcup_{m\geq 0} d_{m,3} \ \sqsubseteq \ \ldots$$

Moreover

$$\bigsqcup_{m\geq 0} \left(\bigsqcup_{n\geq 0} d_{m,n} \right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0} \left(\bigsqcup_{m\geq 0} d_{m,n} \right) .$$

$f(\bigsqcup_n d_n) \sqsubseteq \bigsqcup_n (f d_n)$

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n\geq 0} d_n) = \bigsqcup_{n\geq 0} f(d_n) \quad \text{in } E.$$

$f(\bigsqcup_n d_n) \sqsupseteq \bigsqcup_n f(d_n)$

$f : D \to E$

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \bigsqcup_n d_n$

$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots \sqsubseteq f(d_n) \sqsubseteq \cdots$

log(1)

Continuity and strictness

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  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that
  $$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\perp) = \perp$.

$$\perp \sqsubseteq f(\perp) \sqsubseteq f(f\perp) \sqsubseteq \cdots \sqsubseteq f^n(\perp) \sqsubseteq \cdots \sqsubseteq \bigsqcup_n f^n(\perp) \overset{\text{def}}{=} fix(f)$$

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$fix(f) = \bigsqcup_{n \geq 0} f^n(\perp).$$

is a least pre fixed point of f

- Moreover, $fix(f)$ is a fixed point of f, *i.e.* satisfies $f\big(fix(f)\big) = fix(f)$, and hence is the least fixed point of f.

$\Downarrow$

f has a fixed point

$[\![\textbf{while } B \textbf{ do } C]\!]$: States $\to$ States

$[\![\textbf{while } B \textbf{ do } C]\!]$

$= \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$

$= \bigsqcup_{n \geq 0} {f_{[\![B]\!],[\![C]\!]}}^{n}(\bot)$

$f_{[\![B]\!],[\![C]\!]} : (\text{States} \to \text{States}) \to (\text{State} \to \text{State})$

continuous.

$$= \quad \lambda s \in \mathit{State}. \begin{cases} [\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}$$