# ***Topic 7***

Relating Denotational and Operational Semantics

# Adequacy

For any closed PCF terms $M$ and $V$ of *ground* type $\gamma \in \{nat, bool\}$ with $V$ a value

$$[\![M]\!] = [\![V]\!] \in [\![\gamma]\!] \implies M \Downarrow_\gamma V \, .$$

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but

$$\mathbf{fn}\, x : \tau .\, (\mathbf{fn}\, y : \tau .\, y)\, x \not\Downarrow_{\tau \to \tau} \mathbf{fn}\, x : \tau .\, x$$

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# Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ► Consider $M$ to be $M_1\, M_2$, $\mathbf{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.

   This statement roughly takes the form:

   $$\boxed{[\![M]\!] \lhd_\tau M \text{ for all types } \tau \text{ and all } M \in \mathrm{PCF}_\tau}$$

   where the *formal approximation relations*

   $$\lhd_\tau \subseteq [\![\tau]\!] \times \mathrm{PCF}_\tau$$

   are *logically* chosen to allow a proof by induction.

## Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{nat, bool\}$,

$$[\![M]\!] \lhd_\gamma M \text{ implies } \underbrace{\forall V\ ([\![M]\!] = [\![V]\!] \implies M \Downarrow_\gamma V)}_{\text{adequacy}}$$

## Definition of $d \lhd_\gamma M \;\; (d \in [\![\gamma]\!], M \in \mathrm{PCF}_\gamma)$ for $\gamma \in \{nat, bool\}$

$$n \lhd_{nat} M \quad \overset{\text{def}}{\Leftrightarrow} \quad \big(n \in \mathbb{N} \;\Rightarrow\; M \Downarrow_{nat} \mathbf{succ}^n(\mathbf{0})\big)$$

$$\begin{aligned} b \lhd_{bool} M \quad \overset{\text{def}}{\Leftrightarrow} \quad & (b = true \;\Rightarrow\; M \Downarrow_{bool} \mathbf{true}) \\ & \& \; (b = false \;\Rightarrow\; M \Downarrow_{bool} \mathbf{false}) \end{aligned}$$

## Proof of: $[\![M]\!] \lhd_\gamma M$ implies adequacy

**Case** $\gamma = nat$.

$$
\begin{aligned}
& [\![M]\!] = [\![V]\!] \\
& \quad \Longrightarrow \ [\![M]\!] = [\![\mathbf{succ}^n(\mathbf{0})]\!] && \text{for some } n \in \mathbb{N} \\
& \quad \Longrightarrow \ n = [\![M]\!] \lhd_\gamma M \\
& \quad \Longrightarrow \ M \Downarrow \mathbf{succ}^n(\mathbf{0}) && \text{by definition of } \lhd_{nat}
\end{aligned}
$$

**Case** $\gamma = bool$ is similar.

## Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

- Consider the case $M = M_1\, M_2$.

$\rightsquigarrow$ *logical* definition

## Definition of

$$f \lhd_{\tau \to \tau'} M \ \left(f \in (\llbracket \tau \rrbracket \to \llbracket \tau' \rrbracket), M \in \mathrm{PCF}_{\tau \to \tau'}\right)$$

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$$
\begin{aligned}
& f \lhd_{\tau\to\tau'} M \\
& \quad \overset{\text{def}}{\Leftrightarrow} \;\; \forall\, x \in \llbracket\tau\rrbracket, N \in \mathrm{PCF}_{\tau} \\
& \qquad\qquad\qquad (x \lhd_{\tau} N \;\Rightarrow\; f(x) \lhd_{\tau'} M\, N)
\end{aligned}
$$

## Requirements on the formal approximation relations, III

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fix}(M')$.

$\leadsto$ *admissibility* property

## Admissibility property

**Lemma.** *For all types $\tau$ and $M \in \mathrm{PCF}_\tau$, the set*

$$\{ d \in [\![\tau]\!] \mid d \lhd_\tau M \}$$

*is an admissible subset of $[\![\tau]\!]$.*

## Further properties

**Lemma.** *For all types* $\tau$, *elements* $d, d' \in [\![\tau]\!]$, *and terms* $M, N, V \in \mathrm{PCF}_\tau$,

1. *If* $d \sqsubseteq d'$ *and* $d' \lhd_\tau M$ *then* $d \lhd_\tau M$.
2. *If* $d \lhd_\tau M$ *and* $\forall V\,(M \Downarrow_\tau V \implies N \Downarrow_\tau V)$ *then* $d \lhd_\tau N$ .

## Requirements on the formal approximation relations, IV

We want to be able to proceed by induction.

- Consider the case $M = \mathbf{fn}\, x : \tau \,.\, M'$.

  $\rightsquigarrow$ *substitutivity* property for open terms

## Fundamental property

**Theorem.** *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_{\tau} M[M_1/x_1, \ldots, M_n/x_n]$ .

# Fundamental property

**Theorem.** *For all* $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ *and all* $\Gamma \vdash M : \tau$, *if* $d_1 \lhd_{\tau_1} M_1, \ldots, d_n \lhd_{\tau_n} M_n$ *then* $[\![\Gamma \vdash M]\!][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \lhd_\tau M[M_1/x_1, \ldots, M_n/x_n]$.

**NB.** The case $\Gamma = \emptyset$ reduces to

$$[\![M]\!] \lhd_\tau M$$

for all $M \in \mathrm{PCF}_\tau$.

## Fundamental property of the relations $\lhd_\tau$

**Proposition.** *If* $\Gamma \vdash M : \tau$ *is a valid PCF typing, then for all* $\Gamma$*-environments* $\rho$ *and all* $\Gamma$*-substitutions* $\sigma$

$$\rho \lhd_\Gamma \sigma \;\Rightarrow\; [\![\Gamma \vdash M]\!](\rho) \lhd_\tau M[\sigma]$$

- $\rho \lhd_\Gamma \sigma$ means that $\rho(x) \lhd_{\Gamma(x)} \sigma(x)$ holds for each $x \in dom(\Gamma)$.
- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for $x$ in $M$, each $x \in dom(\Gamma)$.

## Contextual preorder between PCF terms

Given PCF terms $M_1, M_2$, PCF type $\tau$, and a type environment $\Gamma$, the relation $\boxed{\Gamma \vdash M_1 \leq_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type $\gamma$, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V \in \mathrm{PCF}_\gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \implies \mathcal{C}[M_2] \Downarrow_\gamma V \ .$$

# Extensionality properties of $\leq_{\text{ctx}}$

**At a ground type $\gamma \in \{bool, nat\}$,**

$M_1 \leq_{\text{ctx}} M_2 : \gamma$ holds if and only if

$$\forall V \in \mathrm{PCF}_\gamma \; (M_1 \Downarrow_\gamma V \implies M_2 \Downarrow_\gamma V) \ .$$

**At a function type $\tau \to \tau'$,**

$M_1 \leq_{\text{ctx}} M_2 : \tau \to \tau'$ holds if and only if

$$\forall M \in \mathrm{PCF}_\tau \; (M_1 \, M \leq_{\text{ctx}} M_2 \, M : \tau') \ .$$