Topic 4

Scott Induction
Scott’s Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain $D$.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of $f$ is in $S$, i.e. that

$$\text{fix}(f) \in S,$$

it suffices to prove

$$\forall d \in D \ (d \in S \Rightarrow f(d) \in S).$$
Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 . \, d_n \in S) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$. 
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If $D$ is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of $D$ and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called chain-closed (resp. admissible) iff $\{d \in D \mid \Phi(d)\}$ is a chain-closed (resp. admissible) subset of $D$. 
Let $D$, $E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset

$$\downarrow(d) \overset{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

of $D$ is chain-closed.
Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset
  \[
  \downarrow(d) \overset{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}
  \]
  of $D$ is chain-closed.

- The subsets
  \[
  \{(x, y) \in D \times D \mid x \sqsubseteq y\}
  \]
  and
  \[
  \{(x, y) \in D \times D \mid x = y\}
  \]
  of $D \times D$ are chain-closed.
Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \rightarrow D$ be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$
Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \rightarrow D$ be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$

**Proof by Scott induction.**

Let $d \in D$ be a pre-fixed point of $f$. Then,

$$x \in \downarrow(d) \implies x \sqsubseteq d$$

$$\implies f(x) \sqsubseteq f(d)$$

$$\implies f(x) \sqsubseteq d$$

$$\implies f(x) \in \downarrow(d)$$

Hence,

$$\text{fix}(f) \in \downarrow(d).$$
Building chain-closed subsets (II)

Inverse image:
Let $f : D \to E$ be a continuous function.
If $S$ is a chain-closed subset of $E$ then the inverse image

$$f^{-1}S = \{ x \in D \mid f(x) \in S \}$$

is an chain-closed subset of $D$. 
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv (f(x) \sqsubseteq g(x))$ of $D$.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\text{fix}(g)) \sqsubseteq g(\text{fix}(g)).$$
Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of $D$ then $S \cup T$ and $S \cap T$ are chain-closed subsets of $D$.

- If $\{S_i\}_{i \in I}$ is a family of chain-closed subsets of $D$ indexed by a set $I$, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of $D$.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D. \ P(x, y)$ determines a chain-closed subset of $E$. 
Example (III): Partial correctness

Let $\mathcal{F} : \text{State} \rightarrow \text{State}$ be the denotation of

$$\textbf{while } X > 0 \textbf{ do } (Y := X \ast Y; X := X - 1) .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$

$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y].$$
Recall that

$$\mathcal{F} = \text{fix}(f)$$

where $f : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State})$ is given by

$$f(w) = \lambda(x, y) \in \text{State.} \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$
Proof by Scott induction.

We consider the admissible subset of \((\text{State} \rightarrow \text{State})\) given by

\[
S = \left\{ w \mid \forall x, y \geq 0. \begin{align*}
\text{w[X} & \mapsto x, Y \mapsto y] \downarrow \\
\Rightarrow w[X \mapsto x, Y \mapsto y] & = [X \mapsto 0, Y \mapsto x! \cdot y]
\end{align*} \right\}
\]

and show that

\[
w \in S \implies f(w) \in S.
\]