Complexity Theory

Lecture 6

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http://www.cl.cam.ac.uk/teaching/2223/Complexity
Independent Set

Given a graph \( G = (V, E) \), a subset \( X \subseteq V \) of the vertices is said to be an *independent set*, if there are no edges \((u, v)\) for \( u, v \in X\).

The natural algorithmic problem is, given a graph, find the largest independent set.

To turn this *optimisation problem* into a *decision problem*, we define IND as:

*The set of pairs* \((G, K)\), *where* \( G \) *is a graph, and* \( K \) *is an integer, such that* \( G \) *contains an independent set with* \( K \) *or more vertices.*

IND is clearly in NP. We now show it is NP-complete.
Reduction

We can construct a reduction from 3SAT to IND.

A Boolean expression $\phi$ in 3CNF with $m$ clauses is mapped by the reduction to the pair $(G, m)$, where $G$ is the graph obtained from $\phi$ as follows:

- $G$ contains $m$ triangles, one for each clause of $\phi$, with each node representing one of the literals in the clause.
- Additionally, there is an edge between two nodes in different triangles if they represent literals where one is the negation of the other.
Example

$$(x_1 \lor x_2 \lor \neg x_3) \land (x_3 \lor \neg x_2 \lor \neg x_1)$$
Clique

Given a graph \( G = (V, E) \), a subset \( X \subseteq V \) of the vertices is called a \textit{clique}, if for every \( u, v \in X \), \((u, v)\) is an edge.

As with \textit{IND}, we can define a decision problem:
\textbf{CLIQUE} is defined as:

\textit{The set of pairs} \((G, K)\), \textit{where} \( G \) \textit{is a graph, and} \( K \) \textit{is an integer, such that} \( G \) \textit{contains a clique with} \( K \) \textit{or more vertices}. 
CLIQUE is in NP by the algorithm which guesses a clique and then verifies it.

CLIQUE is NP-complete, since $\text{IND} \leq_p \text{CLIQUE}$ by the reduction that maps the pair $(G, K)$ to $(\bar{G}, K)$, where $\bar{G}$ is the complement graph of $G$. 
**k-Colourability**

A graph $G = (V, E)$ is $k$-colourable, if there is a function

$$\chi : V \rightarrow \{1, \ldots, k\}$$

such that, for each $u, v \in V$, if $(u, v) \in E,$

$$\chi(u) \neq \chi(v)$$

This gives rise to a decision problem for each $k$.  
2-colourability is in P.  
For all $k > 2$, $k$-colourability is NP-complete.
3-Colourability

3-Colourability is in NP, as we can guess a colouring and verify it.

To show NP-completeness, we can construct a reduction from 3SAT to 3-Colourability.

For each variable $x$, we have two vertices $x$, $\bar{x}$ which are connected in a triangle with the vertex $a$ (common to all variables).

In addition, for each clause containing the literals $l_1$, $l_2$ and $l_3$ we have a gadget.
Gadget

With a further edge from \( a \) to \( b \).
Recall the definition of $\text{HAM}$—the language of Hamiltonian graphs.

Given a graph $G = (V, E)$, a $\text{Hamiltonian cycle}$ in $G$ is a path in the graph, starting and ending at the same node, such that every node in $V$ appears on the cycle $\text{exactly once}$.

A graph is called $\text{Hamiltonian}$ if it contains a Hamiltonian cycle.

The language $\text{HAM}$ is the set of encodings of Hamiltonian graphs.
Hamiltonian Cycle

We can construct a reduction from 3SAT to HAM. Essentially, this involves coding up a Boolean expression as a graph, so that every satisfying truth assignment to the expression corresponds to a Hamiltonian circuit of the graph.

This reduction is much more intricate than the one for IND.
Travelling Salesman

Recall the travelling salesman problem

Given

- \( V \) — a set of nodes.
- \( c : V \times V \to \mathbb{IN} \) — a cost matrix.

Find an ordering \( v_1, \ldots, v_n \) of \( V \) for which the total cost:

\[
c(v_n, v_1) + \sum_{i=1}^{n-1} c(v_i, v_{i+1})
\]

is the smallest possible.
Travelling Salesman

As with other optimisation problems, we can make a decision problem version of the Travelling Salesman problem.

The problem TSP consists of the set of triples

\[(V, c : V \times V \to \mathbb{N}, t)\]

such that there is a tour of the set of vertices \(V\), which under the cost matrix \(c\), has cost \(t\) or less.
Reduction

There is a simple reduction from HAM to TSP, mapping a graph \((V, E)\) to the triple \((V, c : V \times V \to \mathbb{N}, n)\), where

\[
c(u, v) = \begin{cases} 
1 & \text{if } (u, v) \in E \\
2 & \text{otherwise}
\end{cases}
\]

and \(n\) is the size of \(V\).