A circuit is a directed graph $G = (V, E)$, with $V = \{1, \ldots, n\}$ together with a labeling: $l : V \rightarrow \{\text{true, false, } \wedge, \vee, \neg\}$, satisfying:

- If there is an edge $(i, j)$, then $i < j$;
- Every node in $V$ has *indegree* at most 2.
- A node $v$ has
  - indegree 0 iff $l(v) \in \{\text{true, false}\}$;
  - indegree 1 iff $l(v) = \neg$;
  - indegree 2 iff $l(v) \in \{\vee, \wedge\}$

The value of the expression is given by the value at node $n$. 
A circuit is a more compact way of representing a Boolean expression.

*Identical subexpressions need not be repeated.*

CVP - the *circuit value problem* is, given a circuit, determine the value of the result node $n$.

CVP is solvable in polynomial time, by the algorithm which examines the nodes in increasing order, assigning a value *true* or *false* to each node.

CVP is complete for $P$ under $L$ reductions. That is, for every language $A$ in $P$,

$$A \leq_L CVP$$
Similarly, it can be shown that Reachability is, in fact, \( \text{NL} \)-complete.

*For any language \( A \in \text{NL} \), we have \( A \leq_L \text{Reachability} \)

\[ L = \text{NL} \text{ if, and only if, Reachability } \in L \]

*Note:* it is known that the reachability problem for *undirected* graphs is in \( L \).
Our aim now is to show that there are languages (or, equivalently, decision problems) that we can prove are not in P.

This is done by showing that, for every reasonable function $f$, there is a language that is not in $\text{TIME}(f)$.

The proof is based on the diagonal method, as in the proof of the undecidability of the halting problem.
Time Hierarchy Theorem

For any constructible function $f$, with $f(n) \geq n$, define the $f$-bounded halting language to be:

$$H_f = \{ [M], x \mid M \text{ accepts } x \text{ in } f(|x|) \text{ steps} \}$$

where $[M]$ is a description of $M$ in some fixed encoding scheme. Then, we can show

$H_f \in \text{TIME}(f(n)^2)$ and $H_f \not\in \text{TIME}(f(|n/2|))$

Time Hierarchy Theorem

For any constructible function $f(n) \geq n$, $\text{TIME}(f(n))$ is properly contained in $\text{TIME}(f(2n + 1)^2)$. 

Strong Hierarchy Theorems

For any constructible function $f(n) \geq n$, $\text{TIME}(f(n))$ is properly contained in $\text{TIME}(f(n)(\log f(n)))$.

**Space Hierarchy Theorem**
For any pair of constructible functions $f$ and $g$, with $f = O(g)$ and $g \neq O(f)$, there is a language in $\text{SPACE}(g(n))$ that is not in $\text{SPACE}(f(n))$.

Similar results can be established for nondeterministic time and space classes.
Consequences

• For each $k$, $\text{TIME}(n^k) \neq \text{P}$.

• $\text{P} \neq \text{EXP}$.

• Any language that is $\text{EXP}$-complete is not in $\text{P}$.

• $\text{L} \neq \text{PSPACE}$.

• $\text{NL} \neq \text{PSPACE}$.

• There are no problems in $\text{P}$ that are complete under linear time reductions.
The End

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