Complexity Theory

Lecture 11

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http://www.cl.cam.ac.uk/teaching/2223/Complexity

Configuration Graph

Define the *configuration graph* of M, x to be the graph whose nodes are the possible configurations, and there is an edge from i to j if, and only if, $i \to_M j$.

Then, M accepts x if, and only if, some accepting configuration is reachable from the starting configuration $(s, \triangleright, x, \triangleright, \varepsilon)$ in the configuration graph of M, x.

Using the $O(n^2)$ algorithm for Reachability, we get that $L(M)$—the language accepted by M—can be decided by a deterministic machine operating in time

$$c'(nc^{f(n)})^2 \sim c'c^{2(\log n + f(n))} \sim k^{(\log n + f(n))}$$

In particular, this establishes that NL $\subseteq$ P and NPSPACE $\subseteq$ EXP.

NL Reachability

We can construct an algorithm to show that the Reachability problem is in NL:

1. write the index of node a in the work space;
2. if i is the index currently written on the work space:
   2.1 if $i = b$ then accept, else
   guess an index j ($\log n$ bits) and write it on the work space.
   2.2 if (i, j) is not an edge, reject, else replace i by j and return to (2).

Savitch's Theorem

Further simulation results for nondeterministic space are obtained by other algorithms for Reachability.

We can show that Reachability can be solved by a *deterministic* algorithm in $O((\log n)^2)$ space.

Consider the following recursive algorithm for determining whether there is a path from a to b of length at most i.

$O((\log n)^2)$ space Reachability algorithm:

Path(a, b, i)
if $i = 1$ and $a \neq b$ and (a, b) is not an edge reject
else if (a, b) is an edge or $a = b$ accept
else, for each node x, check:

1. Path$(a, x, \lfloor i/2 \rfloor)$
2. Path$(x, b, \lceil i/2 \rceil)$

if such an x is found, then accept, else reject.

The maximum depth of recursion is $\log n$, and the number of bits of information kept at each stage is $3 \log n$.

Savitch's Theorem

The space efficient algorithm for reachability used on the configuration graph of a nondeterministic machine shows:

$$\mathsf{NSPACE}(f) \subseteq \mathsf{SPACE}(f^2)$$

for $f(n) \geq \log n$.

This yields

$$\mathsf{PSPACE} = \mathsf{NPSPACE} = \text{co-}\mathsf{NPSPACE}.$$

Complementation

A still more clever algorithm for Reachability has been used to show that nondeterministic space classes are closed under complementation:

If $f(n) \geq \log n$, then

$$\mathsf{NSPACE}(f) = \text{co-}\mathsf{NSPACE}(f)$$

In particular

$$\mathsf{NL} = \text{co-}\mathsf{NL}.$$

Logarithmic Space Reductions

We write

$$A \leq_L B$$

if there is a reduction f of A to B that is computable by a deterministic Turing machine using $O(\log n)$ workspace (with a *read-only* input tape and *write-only* output tape).

Note: We can compose $\leq_L$ reductions. So,

$$\text{if } A \leq_L B \text{ and } B \leq_L C \text{ then } A \leq_L C$$

NP-complete Problems

Analysing carefully the reductions we constructed in our proofs of NP-completeness, we can see that SAT and the various other NP-complete problems are actually complete under $\leq_L$ reductions.

Thus, if SAT $\leq_L A$ for some problem A in L then not only P $=$ NP but also L $=$ NP.

P-complete Problems

It makes little sense to talk of complete problems for the class P with respect to polynomial time reducibility $\leq_P$.

There are problems that are complete for P with respect to *logarithmic space* reductions $\leq_L$.
One example is CVP—the circuit value problem.

That is, for every language A in P,

$$A \leq_L \text{CVP}$$

- If CVP $\in$ L then L $=$ P.
- If CVP $\in$ NL then NL $=$ P.