

# Complexity Theory

Easter 2023

## Suggested Exercises 3

1. Show that a language  $L$  is in **co-NP** if, and only if, there is a nondeterministic Turing machine  $M$  and a polynomial  $p$  such that  $M$  halts in time  $p(n)$  for all inputs of length  $x$ , and  $L$  is exactly the set of strings  $x$  such that *all* computations of  $M$  on input  $x$  end in an accepting state.
2. Define a *strong* nondeterministic Turing machine as one where each computation has three possible outcomes: accept, reject or maybe. If  $M$  is such a machine, we say that it accepts  $L$ , if for every  $x \in L$ , every computation path of  $M$  on  $x$  ends in either accept or maybe, with at least one accept *and* for  $x \notin L$ , every computation path of  $M$  on  $x$  ends in reject or maybe, with at least one such path ending in reject.

Show that if  $L$  is decided by a strong nondeterministic Turing machine running in polynomial time, then  $L \in \text{NP} \cap \text{co-NP}$ .

3. We saw in the lectures that if there is a one-way function, then there is a language  $L$  in **UP** that is not in **P**. Suppose that the **RSA** function described in the lecture notes (page 38) is a one-way function. What is the language  $L$  that can then be proved to be in  $\text{UP} \setminus \text{P}$ ?
4. Consider the algorithm presented in the lecture which establishes that **Reachability** is in  $\text{SPACE}((\log n)^2)$ . What is the time complexity of this algorithm?

Can you generalise the time bound to the entire complexity class? That is, give a class of functions  $F$ , such that

$$\text{SPACE}((\log n)^2) \subseteq \bigcup_{f \in F} \text{TIME}(f)$$

5. Show that, for every nondeterministic machine  $M$  which uses  $O(\log n)$  work space, there is a machine  $R$  with three tapes (**input**, **work** and **output**) which works as follows. On input  $x$ ,  $R$  produces on its output tape a description of the configuration graph for  $M, x$ , and  $R$  uses  $O(\log |x|)$  space on its work tape.

Explain why this means that if **Reachability** is in **L**, then  $\text{L} = \text{NL}$ .

6. Consider the language  $L$  in the alphabet  $\{a, b\}$  given by  $L = \{a^n b^n \mid n \in \mathbb{N}\}$ .  $L \notin \text{SPACE}(c)$  for any constant  $c$ . Why?

7. On page 42 of the notes, a number of functions are listed as being constructible. Show that this is the case by giving, for each one, a description of an appropriate Turing machine.

Prove that if  $f$  and  $g$  are constructible functions and  $f(n) \geq n$ , then so are  $f(g)$ ,  $f + g$ ,  $f \cdot g$  and  $2^f$ .

8. For any constructible function  $f$ , and any language  $L \in \text{NTIME}(f(n))$ , there is a nondeterministic machine  $M$  that accepts  $L$  and all of whose computations terminate in time  $O(f(n))$  for all inputs of length  $n$ . Give a detailed argument for this statement, describing how  $M$  might be obtained from a machine accepting  $L$  in time  $f(n)$ .
9. In the lecture, a proof of the Time Hierarchy Theorem was sketched. Give a similar argument for the following Space Hierarchy Theorem:

**Space Hierarchy.** For every constructible function  $f$ , there is a language in  $\text{SPACE}(f(n) \cdot \log f(n))$  that is not in  $\text{SPACE}(f(n))$ .

10. Show that, if  $\text{SPACE}((\log n)^2) \subseteq \text{P}$ , then  $\text{L} \neq \text{P}$ . (Hint: use the Space Hierarchy Theorem from Exercise 9 above.)
11. **POLYLOGSPACE** is the complexity class

$$\bigcup_k \text{SPACE}((\log n)^k).$$

- (a) Show that, for any  $k$ , if  $A \in \text{SPACE}((\log n)^k)$  and  $B \leq_L A$ , then  $B \in \text{SPACE}((\log n)^k)$ .
- (b) Show that there are no **POLYLOGSPACE**-complete problems with respect to  $\leq_L$ . (Hint: use (a) and the space hierarchy theorem).
- (c) Which of the following might be true:  $\text{P} \subseteq \text{POLYLOGSPACE}$ ,  $\text{P} \supseteq \text{POLYLOGSPACE}$ ,  $\text{P} = \text{POLYLOGSPACE}$ ?
- (d) What is the relationship between the class **POLYLOGSPACE** and the class **Quasi-P** (see Exercise Sheet 1, Question 7)?