Definition. A register machine is specified by:

- finitely many registers $R_0, R_1, \ldots, R_n$ (each capable of storing a natural number);
- a program consisting of a finite list of instructions of the form $\text{label : body}$, where for $i = 0, 1, 2, \ldots$, the $(i + 1)^{\text{th}}$ instruction has label $L_i$.

Instruction body takes one of three forms:

<table>
<thead>
<tr>
<th>$R^+ \rightarrow L'$</th>
<th>add 1 to contents of register $R$ and jump to instruction labelled $L'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^- \rightarrow L', L''$</td>
<td>if contents of $R$ is $&gt; 0$, then subtract 1 from it and jump to $L'$, else jump to $L''$</td>
</tr>
<tr>
<td>HALT</td>
<td>stop executing instructions</td>
</tr>
</tbody>
</table>
Recall: Computable functions

Definition. $f \in \mathbb{N}^n \rightarrow \mathbb{N}$ is (register machine) computable if there is a register machine $M$ with at least $n + 1$ registers $R_0, R_1, \ldots, R_n$ (and maybe more) such that for all $(x_1, \ldots, x_n) \in \mathbb{N}^n$ and all $y \in \mathbb{N}$, the computation of $M$ starting with $R_0 = 0$, $R_1 = x_1$, $\ldots$, $R_n = x_n$ and all other registers set to 0, halts with $R_0 = y$ if and only if $f(x_1, \ldots, x_n) = y$.

N.B. there may be many different $M$ that compute the same partial function $f$. 
Coding programs as numbers
Turing/Church solution of the Entscheidungsproblem uses the idea that (formal descriptions of) algorithms can be the data on which algorithms act.

To realize this idea with Register Machines we have to be able to code RM programs as numbers. (In general, such codings are often called Gödel numberings.)
"Effective" numerical codes

RM program \( \rightarrow \) initial contents of \( R_1, \ldots, R_n \)

\( \text{Prog}, \, [x_1, \ldots, x_n] \) \( \rightarrow \) \( y \)

final contents of \( R_0 \) (if halts)

run the RM
"Effective" numerical codes

\[ \text{Prog} , \left[ x_1, \ldots, x_n \right] \rightarrow y \]

\[ \text{code} \downarrow \uparrow \text{decode} \]

\[ \left< \text{Prog} , [x_1, \ldots, x_n] \right> \]

\[ \text{a number} \]

\[ \text{Want numerical codings} \]
\[ \left< - , - \right>, \left[ - , - \right], \left[ - , - , - \right] \]

\[ \text{So that} \]
\[ \text{decode} \rightarrow \text{run} \rightarrow \text{is RM computable} \]
Numerical coding of pairs

For \( x, y \in \mathbb{N} \), define

\[
\begin{align*}
\langle x, y \rangle & \triangleq 2^x (2y + 1) \\
\langle x, y \rangle & \triangleq 2^x (2y + 1) - 1
\end{align*}
\]

Left-hand side is equal to the right-hand side by definition.
Numerical coding of pairs

For $x, y \in \mathbb{N}$, define

$$\begin{align*}
\langle x, y \rangle &\triangleq 2^x (2y + 1) \\
\langle x, y \rangle &\triangleq 2^x (2y + 1) - 1
\end{align*}$$

<table>
<thead>
<tr>
<th>$\langle x, y \rangle$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>...</td>
</tr>
<tr>
<td>$1$</td>
<td>2</td>
<td>6</td>
<td>10</td>
<td>...</td>
</tr>
<tr>
<td>$2$</td>
<td>4</td>
<td>12</td>
<td>20</td>
<td>...</td>
</tr>
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<td>1</td>
<td>5</td>
<td>9</td>
<td>...</td>
</tr>
<tr>
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<td>3</td>
<td>11</td>
<td>19</td>
<td>...</td>
</tr>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Numerical coding of pairs

For $x, y \in \mathbb{N}$, define
\[
\begin{cases} 
\langle x, y \rangle \triangleq 2^x(2y + 1) \\
\{ x, y \rangle \triangleq 2^x(2y + 1) - 1
\end{cases}
\]

So
\[
\begin{array}{c}
0_b \langle x, y \rangle = 0_{by} 1 0 \ldots 0 \\
0_b \{ x, y \rangle = 0_{by} 0 1 \ldots 1 \\
\end{array}
\]

(Notation: $0_b x \triangleq x$ in binary.)

E.g. $27 = 0_b 11011 = \langle 0, 13 \rangle = \langle 2, 3 \rangle$
Numerical coding of pairs

For \( x, y \in \mathbb{N} \), define
\[
\langle x, y \rangle \triangleq 2^x(2y + 1)
\]
\[
\langle x, y \rangle \triangleq 2^x(2y + 1) - 1
\]

So
\[
\begin{align*}
0b \langle x, y \rangle &= 0by10\ldots0 \\
0b \langle x, y \rangle &= 0by01\ldots1
\end{align*}
\]

\( \langle -, - \rangle \) gives a bijection (one-one correspondence) between \( \mathbb{N} \times \mathbb{N} \) and \( \mathbb{N} \).

\( \langle -, - \rangle \) gives a bijection between \( \mathbb{N} \times \mathbb{N} \) and \( \{ n \in \mathbb{N} \mid n \neq 0 \} \).
Numerical coding of lists

\textit{list} \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using ML notation for lists:}

\begin{itemize}
  \item empty list: \([]\)
  \item list-cons: \(x :: l \in \text{list} \mathbb{N}\) (given \(x \in \mathbb{N}\) and \(l \in \text{list} \mathbb{N}\))
  \item \([x_1, x_2, \ldots, x_n] \triangleq x_1 :: (x_2 :: (\cdots x_n :: [] \cdots ))\)
\end{itemize}
**Numerical coding of lists**

\( \text{list } \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using ML notation for lists.} \)

For \( \ell \in \text{list } \mathbb{N} \), define \( \overline{\ell} \in \mathbb{N} \) by induction on the length of the list \( \ell \):

\[
\begin{align*}
\overline{[]} & \triangleq 0 \\
\overline{x :: \ell} & \triangleq \langle x, \overline{\ell} \rangle = 2^x (2 \cdot \overline{\ell} + 1)
\end{align*}
\]

Thus \( \overline{[x_1, x_2, \ldots, x_n]} = \langle x_1, \langle x_2, \ldots \langle x_n, 0 \rangle \ldots \rangle \rangle \)
Numerical coding of lists

\( \text{list } \mathbb{N} \triangleq \) set of all finite lists of natural numbers, using ML notation for lists.

For \( \ell \in \text{list } \mathbb{N} \), define \( \llbracket \ell \rrbracket \in \mathbb{N} \) by induction on the length of the list \( \ell \):

\[
\begin{align*}
\llbracket \ [] \rrbracket & \triangleq 0 \\
\llbracket x :: \ell \rrbracket & \triangleq \langle x, \llbracket \ell \rrbracket \rangle = 2^x (2 \cdot \llbracket \ell \rrbracket + 1)
\end{align*}
\]

For example:

\( \llbracket [3] \rrbracket = \llbracket 3 :: [] \rrbracket = \langle 3, 0 \rangle = 2^3 (2 \cdot 0 + 1) = 8 = \text{0b}1000 \)

\( \llbracket [1, 3] \rrbracket = \langle 1, \llbracket [3] \rrbracket \rangle = \langle 1, 8 \rangle = 34 = \text{0b}100010 \)

\( \llbracket [2, 1, 3] \rrbracket = \langle 2, \llbracket [1, 3] \rrbracket \rangle = \langle 2, 34 \rangle = 276 = \text{0b}100010100 \)
Numerical coding of lists

\[ \text{list } \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using ML notation for lists.} \]

For \( l \in \text{list } \mathbb{N} \), define \( \text{\( l \)} \) \( \in \) \( \mathbb{N} \) by induction on the length of the list \( l \):

\[
\begin{align*}
\text{\( [\] } \text{\( \triangleq 0 \)} \\
\text{\( x \cdot l \text{\( \triangleq \)} \text{\( \langle x, \text{\( l \)} \rangle = 2^x (2 \cdot \text{\( l \)} + 1) \)}}
\end{align*}
\]

For example:

\[
\begin{align*}
\text{\( [3] \)} & = \text{\( [3 \cdot [\] \)} = \text{\( \langle 3, 0 \rangle = 2^3 (2 \cdot 0 + 1) = 8 = 0b1000 \)} \\
\text{\( [1, 3] \)} & = \text{\( \langle 1, [3] \rangle \)} = \text{\( \langle 1, 8 \rangle = 34 = 0b100010 \)} \\
\text{\( [2, 1, 3] \)} & = \text{\( \langle 2, [1, 3] \rangle \)} = \text{\( \langle 2, 34 \rangle = 276 = 0b100010100 \)}
\end{align*}
\]
**Numerical coding of lists**

\( \text{list } \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using } \text{ML notation for lists.} \)

For \( \ell \in \text{list } \mathbb{N} \), define \( \langle \ell \rangle \in \mathbb{N} \) by induction on the length of the list \( \ell \):

\[
\begin{cases}
\langle [] \rangle & \triangleq 0 \\
\langle x::\ell \rangle & \triangleq \langle x, \langle \ell \rangle \rangle = 2^x (2 \cdot \langle \ell \rangle + 1)
\end{cases}
\]

\[
\begin{array}{c}
0b^\perp [x_1, x_2, \ldots, x_n]^\perp = \begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
x_n & 0 & \cdots \\
x_{n-1} & 0 & \cdots \\
x_1 & 0 & \cdots \\
\end{bmatrix}
\end{array}
\]
Numerical coding of lists

\( \text{list } \mathbb{N} \triangleq \text{set of all finite lists of natural numbers, using ML notation for lists.} \)

For \( \ell \in \text{list } \mathbb{N} \), define \( \llbracket \ell \rrbracket \in \mathbb{N} \) by induction on the length of the list \( \ell \):

\[
\begin{align*}
\llbracket \[] \rrbracket & \triangleq 0 \\
\llbracket x :: \ell \rrbracket & \triangleq \langle x, \llbracket \ell \rrbracket \rangle = 2^x(2 \cdot \llbracket \ell \rrbracket + 1)
\end{align*}
\]

\[
0b^\llbracket [x_1, x_2, \ldots, x_n] \rrbracket = \begin{array}{cccccc}
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0
\end{array}
\]

Hence \( \ell \mapsto \llbracket \ell \rrbracket \) gives a bijection from \( \text{list } \mathbb{N} \) to \( \mathbb{N} \).
Numerical coding of programs

If $P$ is the RM program

\[ L_0 : body_0 \\
L_1 : body_1 \\
\vdots \\
L_n : body_n \]

then its numerical code is

\[ \llbracket P \rrbracket \triangleq \llbracket [\llbracket body_0 \rrbracket, \ldots, \llbracket body_n \rrbracket] \rrbracket \]

where the numerical code $\llbracket body \rrbracket$ of an instruction body is defined by:

\[ \begin{align*}
\llbracket R_i^+ \rightarrow L_j \rrbracket & \triangleq \llbracket 2i, j \rrbracket \\
\llbracket R_i^- \rightarrow L_j, L_k \rrbracket & \triangleq \llbracket 2i + 1, \langle j, k \rangle \rrbracket \\
\llbracket \text{HALT} \rrbracket & \triangleq 0
\end{align*} \]
Any $x \in \mathbb{N}$ decodes to a unique instruction $\text{body}(x)$:

if $x = 0$ then $\text{body}(x)$ is HALT,
else ($x > 0$ and) let $x = \langle y, z \rangle$ in
  if $y = 2i$ is even, then
    $\text{body}(x)$ is $R_i^+ \to L_z$,
  else $y = 2i + 1$ is odd, let $z = \langle j, k \rangle$ in
    $\text{body}(x)$ is $R_i^- \to L_j, L_k$

So any $e \in \mathbb{N}$ decodes to a unique program $\text{prog}(e)$, called the register machine program with index $e$:

$$
\text{prog}(e) \triangleq \\
\begin{align*}
L_0 &: \text{body}(x_0) \\
& \vdots \\
L_n &: \text{body}(x_n)
\end{align*}
$$

where $e = ^\downarrow [x_0, \ldots, x_n] ^\uparrow$
Example of \( \text{prog}(e) \)

- \( 786432 = 2^{19} + 2^{18} = 0b110\ldots0 = [18, 0] \)
  - 18 "0"s

- \( 18 = 0b10010 = \langle 1, 4 \rangle = \langle 1, \langle 0, 2 \rangle \rangle = R_0^{-} \rightarrow L_0, L_2 \)

- \( 0 = \text{HALT}^{-} \)

So \( \text{prog}(786432) = \begin{array}{c}
\text{L}_0 : R_0^{-} \rightarrow L_0, L_2 \\
\text{L}_1 : \text{HALT}
\end{array} \)
Example of \textit{prog}(e)

- \[ 786432 = 2^{19} + 2^{18} = 0b110\ldots0 = \overline{[18, 0]} \]
  - 18 "0"s

- \[ 18 = 0b10010 = \langle 1, 4 \rangle = \langle 1, \langle 0, 2 \rangle \rangle = \overline{R_0^- \rightarrow L_0, L_2} \]

- \[ 0 = \overline{\text{HALT}} \]

So \( \text{prog}(786432) = \)

\[
\begin{array}{c}
\text{L}_0 : R_0^- \rightarrow L_0, L_2 \\
\text{L}_1 : \text{HALT}
\end{array}
\]

N.B. jump to label with no body (erroneous halt)

What function is computed by a RM with \( \text{prog}(786432) \) as its program?
666 = 0b1010011010

= \left[ 1, 1, 0, 2, 1 \right]^\top

\text{prog}(666) =

\begin{align*}
L_0 : & \mathbb{R}_0^+ \rightarrow L_0 \\
L_1 : & \mathbb{R}_0^+ \rightarrow L_0 \\
L_2 : & \text{HALT} \\
L_3 : & \mathbb{R}_0^- \rightarrow L_0, L_0 \\
L_4 : & \mathbb{R}_0^+ \rightarrow L_0
\end{align*}

\text{(never halts!)}

What partial function does this compute?
Example of \textit{prog}(e)

- \(786432 = 2^{19} + 2^{18} = 0b110\ldots0 = \lceil[18, 0]\rceil\), 18 "0"s

- \(18 = 0b10010 = \langle 1, 4 \rangle = \langle 1, \langle 0, 2 \rangle \rangle = \lceil R_0^- \rightarrow L_0, L_2 \rceil\)

- \(0 = \lceil \text{HALT} \rceil\)

So \(\text{prog}(786432) = \begin{cases} L_0 : R_0^- \rightarrow L_0, L_2 \\ L_1 : \text{HALT} \end{cases}\)

\[\begin{array}{c}
\text{N.B. In case } e = 0 \text{ we have } 0 = \lceil \text{[]} \rceil, \text{ so } \text{prog}(0) \text{ is the program with an empty list of instructions, which by convention we regard as a RM that does nothing (i.e. that halts immediately).}
\end{array}\]
"Effective" numerical codes

\[ \text{Prog}, [x_1, \ldots, x_n] \rightarrow y \]

\[ \text{code} \downarrow \quad \uparrow \text{decode} \]

\[ \langle \text{Prog}, [x_1, \ldots, x_n] \rangle \]

- A number

Want numerical codings

\[ \langle -, -, \rangle, \langle -, \rangle, \langle [-, \ldots, -] \rangle \]

So that

- Decode \quad \Rightarrow \quad \text{run}

is RM computable
Universal register machine, $U$
High-level specification

Universal RM $U$ carries out the following computation, starting with $R_0 = 0$, $R_1 = e$ (code of a program), $R_2 = a$ (code of a list of arguments) and all other registers zeroed:

- decode $e$ as a RM program $P$
- decode $a$ as a list of register values $a_1$, \ldots, $a_n$
- carry out the computation of the RM program $P$ starting with $R_0 = 0, R_1 = a_1, \ldots, R_n = a_n$ (and any other registers occurring in $P$ set to 0).
Mnemonics for the registers of $U$ and the role they play in its program:

$R_1 \equiv P$ code of the RM to be simulated

$R_2 \equiv A$ code of current register contents of simulated RM

$R_3 \equiv PC$ program counter—number of the current instruction (counting from 0)

$R_4 \equiv N$ code of the current instruction body

$R_5 \equiv C$ type of the current instruction body

$R_6 \equiv R$ current value of the register to be incremented or decremented by current instruction (if not HALT)

$R_7 \equiv S$, $R_8 \equiv T$ and $R_9 \equiv Z$ are auxiliary registers.
Overall structure of U’s program

1. copy PCth item of list in P to N; goto 2

2. if \( N = 0 \) then copy 0th item of list in A to \( R_0 \) and halt, else (decode \( N \) as \( \langle y, z \rangle \); \( C ::= y; N ::= z \); goto 3)

{at this point either \( C = 2i \) is even and current instruction is \( R_i^+ \rightarrow L_z \),
or \( C = 2i + 1 \) is odd and current instruction is \( R_i^- \rightarrow L_j, L_k \) where \( z = \langle j, k \rangle \)}

3. copy i-th item of list in A to R; goto 4

4. execute current instruction on R; update PC to next label; restore register values to A; goto 1
Overall structure of $U$'s program

1. copy PCth item of list in $P$ to $N$; goto 2

2. if $N = 0$ then copy 0th item of list in $A$ to $R_0$ and halt, else (decode $N$ as $\langle y, z \rangle$; $C ::= y$; $N ::= z$; goto 3)

{at this point either $C = 2i$ is even and current instruction is $R_i^+ \rightarrow L_z$, or $C = 2i + 1$ is odd and current instruction is $R_i^- \rightarrow L_j, L_k$ where $z = \langle j, k \rangle$}

3. copy $i$th item of list in $A$ to $R$; goto 4

4. execute current instruction on $R$; update $PC$ to next label; restore register values to $A$; goto 1

To implement this, we need RMs for manipulating (codes of) lists of numbers...