Lecture 14

Dependent Types

A brief look at some category theory for modelling type theories with dependent types.

Will restrict attention to the case of **Set**, rather than in full generality.

Further reading:
M. Hofmann, *Syntax and Semantics of Dependent Types.* In: A.M. Pitts and P. Dybjer (eds), *Semantics and Logics of Computation* (CUP, 1997).

Simple types

$$\diamond, x_1 : T_1, \ldots, x_n : T_n \vdash t(x_1, \ldots, x_n) : T$$

Dependent types

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and more generally

$$\diamond,\ x_1 : T_1,\ x_2 : T_2(x_1),\ x_3 : T_3(x_1, x_2),\ \ldots \vdash \\ t(x_1, x_2, x_3, \ldots) : T(x_1, x_2, x_3, \ldots)$$

If type expressions denote sets, then

a type $T_1(x)$ dependent upon $x : T$

should denote

an indexed family of sets $(E\, i \mid i \in I)$
(where I is the set denoted by type T)

i.e. $E : I \to \mathbf{Set}$ is a set-valued function on a set I.

For each $I \in \mathbf{Set}$, let $\boxed{\mathbf{Set}^I}$ be the category with

- $\texttt{obj}(\mathbf{Set}^I) \triangleq (\texttt{obj}\,\mathbf{Set})^I$, so objects are I-indexed families of sets, $X = (X_i \mid i \in I)$
- morphisms $f : X \to Y$ in $\mathbf{Set}^I$ are I-indexed families of functions $f = (f_i \in \mathbf{Set}(X_i, Y_i) \mid i \in I)$
- composition: $(g \circ f) \triangleq (g_i \circ f_i \mid i \in I)$

  (i.e. use composition of functions in $\mathbf{Set}$ at each index $i \in I$)
- identity: $\texttt{id}_X \triangleq (\texttt{id}_{X_i} \mid i \in I)$

  (i.e. use identity functions in $\mathbf{Set}$ at each index $i \in I$)

For each $p : I \to J$ in $\mathbf{Set}$, let $\boxed{p^* : \mathbf{Set}^J \to \mathbf{Set}^I}$ be the functor defined by:

$$p^*\left(\begin{array}{c|c} Y_j & \\ \big\downarrow f_j & j \in J \\ Y'_j & \end{array}\right) \triangleq \left(\begin{array}{c|c} Y_{p\,i} & \\ \big\downarrow f_{p\,i} & i \in I \\ Y'_{p\,i} & \end{array}\right)$$

i.e. p^* takes J-indexed families of sets/functions to I-indexed ones by precomposing with p

Dependent products

of families of sets

For $I, J \in \mathbf{Set}$, consider the functor $\pi_1^* : \mathbf{Set}^I \to \mathbf{Set}^{I \times J}$ induced by precomposition with the first projection function $\pi_1 : I \times J \to I$.

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \to \mathbf{Set}^I$.

Proof. We apply the Theorem from Lecture 13: for each $E \in \mathbf{Set}^{I \times J}$ we define $\Sigma\, E \in \mathbf{Set}^I$ and $\eta_E : E \to \pi_1^*(\Sigma\, E)$ in $\mathbf{Set}^{I \times J}$ with the required universal property…

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I \times J} \longrightarrow \mathbf{Set}^I$.

For each $E \in \mathbf{Set}^{I\times J}$, define $\Sigma\, E \in \mathbf{Set}^I$ to be the function mapping each $i \in I$ to the set

$$\boxed{(\Sigma\, E)_i \triangleq \sum_{j \in J} E_{(i,j)} = \{(j, e) \mid j \in J \ \wedge \ e \in E_{(i,j)}\}}$$

Theorem. π_1^* has a left adjoint $\Sigma : \mathbf{Set}^{I\times J} \longrightarrow \mathbf{Set}^I$.

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$$\boxed{(\Sigma\, E)_i \triangleq \textstyle\sum_{j\in J} E_{(i,j)} = \{(j,e) \mid j \in J \;\wedge\; e \in E_{(i,j)}\}}$$

and define $\eta_E : E \longrightarrow \pi_1^*(\Sigma\, E)$ in $\mathbf{Set}^{I\times J}$ to be the function mapping each $(i,j) \in I \times J$ to the function $(\eta_E)_{(i.j)} : E_{(i,j)} \longrightarrow (\Sigma\, E)_i$ given by $\boxed{e \mapsto (j,e)}$.

Universal property–

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Universal property–*existence part*: given any $X \in \mathbf{Set}^I$ and $f : E \longrightarrow \pi_1^*(X)$ in $\mathbf{Set}^{I\times J}$, we have

$$\begin{tikzcd} E \arrow[r, "\eta_E"] \arrow[dr, "f"'] & \pi_1^*(\Sigma\, E) \arrow[d, dashed, "\pi_1^*(\overline{f})"] & \Sigma\, E \arrow[d, dashed, "\overline{f}"] \\ & \pi_1^*(X) & X \end{tikzcd}$$

where for all $i \in I$, $j \in J$ and $e \in E_{(i,j)}$ $\boxed{\overline{f}_i(j,e) \triangleq f_{(i,j)}(e)}$

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Universal property–*uniqueness part*: given $g : \Sigma\, E \longrightarrow X$ in $\mathbf{Set}^I$ making

$$\begin{tikzcd} E \arrow[r, "\eta_E"] \arrow[dr, "f"'] & \pi_1^*(\Sigma\, E) \arrow[d, dashed, "\pi_1^*(g)"] \\ & \pi_1^*(X) \end{tikzcd}$$

commute in $\mathbf{Set}^{I\times J}$,

then for all $i \in I$, and $(j,e) \in (\Sigma\, E)_i$ we have

$$\overline{f}_i(j,e) \triangleq f_{(i,j)}(e) = (\pi_1^* g \circ \eta_E)_{(i,j)}\, e = (\pi_1^* g)_{(i,j)}((\eta_E)_{(i,j)}\, e) \triangleq g_i(j,e)$$

so $g = \overline{f}$. □

Dependent functions

of families of sets

We have seen that the left adjoint to $\pi_1^* : \mathbf{Set}^I \to \mathbf{Set}^{I\times J}$ is given by dependent products of sets.

Dually, dependent function sets give:

Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I\times J} \to \mathbf{Set}^I$.

Proof. We apply the Theorem from Lecture 13: for each $E \in \mathbf{Set}^{I\times J}$ we define $\Pi\, E \in \mathbf{Set}^I$ and $\varepsilon_E : \pi_1^*(\Pi\, E) \to E$ in $\mathbf{Set}^{I\times J}$ with the required universal property…

Theorem. π_1^* has a right adjoint $\Pi : \mathbf{Set}^{I\times J} \longrightarrow \mathbf{Set}^I$.

For each $E \in \mathbf{Set}^{I\times J}$, define $\Pi\, E \in \mathbf{Set}^I$ to be the function mapping each $i \in I$ to the set

$$\boxed{(\Pi\, E)_i \triangleq \prod_{j\in J} E_{(i,j)} = \{f \subseteq (\Sigma\, E)_i \mid f \text{ is single-value and total}\}}$$

where $f \subseteq (\Sigma\, E)_i$ is

single-valued if $\forall j \in J, \forall e, e' \in E_{(i,j)},\ (j,e) \in f \wedge (j,e') \in f \Rightarrow e = e'$

total if $\forall j \in J, \exists e \in E_{(i,j)}\ (j,e) \in f$

Thus each $f \in (\Pi\, E)_i$ is a dependently typed function mapping elements $j \in J$ to elements of $E_{(i,j)}$ (result set depends on the argument j).

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and define $\varepsilon_E : \pi_1^*(\Pi\, E) \longrightarrow E$ in $\mathbf{Set}^{I\times J}$ to be the function mapping each $(i, j) \in I \times J$ to the function $(\varepsilon_E)_{(i,j)} : (\Pi\, E)_i \longrightarrow E_{(i,j)}$ given by $\boxed{f \mapsto f\, j}$ = unique $e \in E_{(i,j)}$ such that $(j, e) \in f$.

Universal property–

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Universal property–*existence part*: given any $X \in \mathbf{Set}^{I}$ and $f : \pi_1^*(X) \longrightarrow E$ in $\mathbf{Set}^{I\times J}$, we have

$$\begin{tikzcd}
\Pi\, E & \pi_1^*(\Pi\, E) \arrow[r, "\varepsilon_E"] & E \\
X \arrow[u, dashed, "\overline{f}"] & \pi_1^*(X) \arrow[u, dashed, "\pi_1^*(\overline{f})"] \arrow[ur, "f"'] &
\end{tikzcd}$$

where for all $i \in I$ and $x \in X_i$ $\boxed{\overline{f}_i\, x \triangleq \{(j, f_{(i,j)}\, x) \mid j \in J\}}$

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Universal property–*uniqueness part*: given $g : X \to \Pi\, E$ in $\mathbf{Set}^I$ making

$$\begin{tikzcd} \pi_1^*(\Pi\, E) \arrow[r, "\varepsilon_E"] & E \\ \pi_1^*(X) \arrow[u, "\pi_1^*(g)"] \arrow[ur, "f"'] & \end{tikzcd}$$

commute in $\mathbf{Set}^{I \times J}$,

then for all $i \in I$, $j \in J$ and $x \in X_i$ we have

$$\overline{f}_i\, x\, j \triangleq f_{(i,j)}\, x = (\varepsilon_E \circ \pi_1^* g)_{(i,j)}\, x = (\varepsilon_E)_{(i,j)}(g_i\, x) \triangleq g_i\, x\, j$$

so $g = \overline{f}$. □

Isomorphism of categories

Two categories $\mathbf{C}$ and $\mathbf{D}$ are isomorphic if they are isomorphic objects in the category of all categories of some given size, that is, if there are functors

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd} \text{ with } \mathtt{id}_{\mathbf{C}} = G \circ F \text{ and } F \circ G = \mathtt{id}_{\mathbf{D}}.$$

In which case, as usual, we write $\boxed{\mathbf{C} \cong \mathbf{D}}$.

Equivalence of categories

Two categories $\mathbf{C}$ and $\mathbf{D}$ are equivalent if there are functors $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$ and natural isomorphisms $\eta : \mathtt{id}_{\mathbf{C}} \cong G \circ F$ and $\varepsilon : F \circ G \cong \mathtt{id}_{\mathbf{D}}$.
In which case, one writes $\boxed{\mathbf{C} \simeq \mathbf{D}}$.

Equivalence of categories

Two categories $\mathbf{C}$ and $\mathbf{D}$ are equivalent if there are functors $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] & \mathbf{D} \arrow[l, bend left, "G"] \end{tikzcd}$ and natural isomorphisms $\eta : \mathrm{id}_{\mathbf{C}} \cong G \circ F$ and $\varepsilon : F \circ G \cong \mathrm{id}_{\mathbf{D}}$.

In which case, one writes $\boxed{\mathbf{C} \simeq \mathbf{D}}$.

Some deep results in mathematics take the form of equivalences of categories. E.g.

Stone duality: $\begin{pmatrix}\text{category of}\\ \text{Boolean algebras}\end{pmatrix}^{\mathrm{op}} \simeq \begin{pmatrix}\text{category of compact}\\ \text{totally disconnected}\\ \text{Hausdorff spaces}\end{pmatrix}$

Gelfand duality: $\begin{pmatrix}\text{category of}\\ \text{abelian } C^* \text{ algebras}\end{pmatrix}^{\mathrm{op}} \simeq \begin{pmatrix}\text{category of compact}\\ \text{Hausdorff spaces}\end{pmatrix}$

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

$\mathbf{Set}/I$ is a slice category:

- objects are pairs (E, p) where $E \in \mathtt{obj}\,\mathbf{Set}$ and $p \in \mathbf{Set}(E, I)$
- morphisms $g : (E, p) \longrightarrow (E', p')$ are $f \in \mathbf{Set}(E, E')$ satisfying $p' \circ f = p$ in $\mathbf{Set}$
- composition and identities – as for $\mathbf{Set}$

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

There are functors $F : \mathbf{Set}^I \longrightarrow \mathbf{Set}/I$ and $G : \mathbf{Set}/I \longrightarrow \mathbf{Set}^I$, given on objects and morphisms by:

$$F\,X \triangleq (\{(i,x) \mid i \in I \wedge x \in X_i\}, \mathtt{fst})$$

$$F\,f\,(i,x) \triangleq (i, f_i\,x)$$

$$G(E,p) \triangleq (\{e \in E \mid p\,e = i\} \mid i \in I)$$

$$(G\,f)_i\,e \triangleq f\,e$$

Example: $\mathbf{Set}^I \simeq \mathbf{Set}/I$

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$$F\,f\,(i,x) \triangleq (i, f_i\,x)$$

$$G(E,p) \triangleq (\{e \in E \mid p\,e = i\} \mid i \in I)$$
$$(G\,f)_i\,e \triangleq f\,e$$

There are natural isomorphisms

$$\eta : \mathtt{id}_{\mathbf{Set}^I} \cong G \circ F \text{ and } \varepsilon : F \circ G \cong \mathtt{id}_{\mathbf{Set}/I}$$

defined by… [exercise]

FACT Given $p : I \longrightarrow J$ in $\mathbf{Set}$, the composition

$$\mathbf{Set}/J \simeq \mathbf{Set}^J \xrightarrow{p^*} \mathbf{Set}^I \simeq \mathbf{Set}/I$$

is the functor "pullback along p".

One can generalize from $\mathbf{Set}$ to any category $\mathbf{C}$ with pullbacks and model Σ/Π types by left/right adjoints to pullback functors – see locally cartesian closed categories in the literature.