Category Theory

Lecture 10

Assessed Exercise Sheet 4 available
(solutions due Fri 4 Nov, 12 noon)

Solution notes for Ex. Sh. 3 available

Curry-Howard correspondence

Logic		**Type Theory**
propositions	$\leftrightarrow$	types
proofs	$\leftrightarrow$	terms

E.g. IPL *versus* STLC.

Curry-Howard-Lawvere/Lambek correspondence

Logic		**Type Theory**		**Category Theory**
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
proofs	$\leftrightarrow$	terms	$\leftrightarrow$	morphisms

E.g. IPL *versus* STLC *versus* CCCs

Curry-Howard-Lawvere/Lambek correspondence

Logic		**Type Theory**		**Category Theory**
propositions	$\leftrightarrow$	types	$\leftrightarrow$	objects
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E.g. IPL *versus* STLC *versus* CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.

Functors

are the appropriate notion of morphism between categories

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $\boxed{F : \mathbf{C} \longrightarrow \mathbf{D}}$ is specified by:

- a function $\mathtt{obj}\,\mathbf{C} \longrightarrow \mathtt{obj}\,\mathbf{D}$ whose value at X is written $\boxed{F\,X}$
- for all $X, Y \in \mathbf{C}$, a function $\mathbf{C}(X, Y) \longrightarrow \mathbf{D}(F\,X, F\,Y)$ whose value at $f : X \longrightarrow Y$ is written $\boxed{F\,f : F\,X \longrightarrow F\,Y}$ and which is required to preserve composition and identity morphisms:

$$\begin{aligned} F(g \circ f) &= F\,g \circ F\,f \\ F(\mathtt{id}_X) &= \mathtt{id}_{F\,X} \end{aligned}$$

Examples of functors

"Forgetful" functors from categories of set-with-structure back to **Set**.

E.g. $\boxed{U : \mathbf{Mon} \longrightarrow \mathbf{Set}}$

$$\begin{cases} U(M, \cdot, e) & = M \\ U((M_1, \cdot_1, e_1) \xrightarrow{f} (M_2, \cdot_2, e_2)) & = M_1 \xrightarrow{f} M_2 \end{cases}$$

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Similarly $U : \mathbf{Preord} \to \mathbf{Set}$.

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \longrightarrow \mathbf{Mon}}$

Given $\Sigma \in \mathbf{Set}$,

$$F\,\Sigma = (\mathtt{List}\,\Sigma, @, \mathtt{nil}), \text{ the free monoid on } \Sigma$$

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Free monoid functor $\boxed{F : \mathbf{Set} \longrightarrow \mathbf{Mon}}$

Given $\Sigma \in \mathbf{Set}$,

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Given a function $f : \Sigma_1 \longrightarrow \Sigma_2$, we get a function $F\,f : \mathtt{List}\,\Sigma_1 \longrightarrow \mathtt{List}\,\Sigma_2$ by mapping f over finite lists:

$$F\,f\,[a_1, \ldots, a_n] = [f\,a_1, \ldots, f\,a_n]$$

This gives a monoid morphism $F\,\Sigma_1 \longrightarrow F\,\Sigma_2$; and mapping over lists preserves composition ($F(g \circ f) = F\,g \circ F\,f$) and identities ($F\,\mathtt{id}_\Sigma = \mathtt{id}_{F\,\Sigma}$). So we do get a functor from **Set** to **Mon**.

Examples of functors

If $\mathbf{C}$ is a category with binary products and $X \in \mathbf{C}$, then the function $(\_) \times X : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $\boxed{(\_) \times X : \mathbf{C} \to \mathbf{C}}$ mapping morphisms $f : Y \to Y'$ to

$$f \times \mathtt{id}_X : Y \times X \to Y' \times X$$

$$\left(\text{recall that } f \times g \text{ is the unique morphism with } \begin{cases} \pi_1 \circ (f \times g) & = f \circ \pi_1 \\ \pi_2 \circ (f \times g) & = g \circ \pi_2 \end{cases}\right)$$

since it is the case that

$$\begin{cases} \mathtt{id}_X \times \mathtt{id}_Y & = \mathtt{id}_{X \times Y} \\ (f' \circ f) \times \mathtt{id}_X & = (f' \times \mathtt{id}_X) \circ (f \times \mathtt{id}_X) \end{cases}$$

(see Exercise Sheet 2, question 1c).

Examples of functors

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $(\_)^X : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $\boxed{(\_)^X : \mathbf{C} \to \mathbf{C}}$ mapping morphisms $f : Y \to Y'$ to

$$f^X \triangleq \mathtt{cur}(f \circ \mathtt{app}) : Y^X \to Y'^X$$

since it is the case that $\begin{cases} (\mathtt{id}_Y)^X & = \mathtt{id}_{Y^X} \\ (g \circ f)^X & = g^X \circ f^X \end{cases}$

(see Exercise Sheet 3, question 4).

Contravariance

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{D}$ is called a contravariant functor from $\mathbf{C}$ to $\mathbf{D}$.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{C}$, then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathbf{C}^{\mathrm{op}}$

so $F\,X \xleftarrow{F\,f} F\,Y \xleftarrow{F\,g} F\,Z$ in $\mathbf{D}$ and hence

$$\boxed{F(g \circ_{\mathbf{C}} f) = F\,f \circ_{\mathbf{D}} F\,g}$$

(contravariant functors reverse the order of composition)

A functor $\mathbf{C} \longrightarrow \mathbf{D}$ is sometimes called a covariant functor from $\mathbf{C}$ to $\mathbf{D}$.

Example of a contravariant functor

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $X^{(\_)} : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $\boxed{X^{(\_)} : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}}$ mapping morphisms $f : Y \to Y'$ to

$$X^f \triangleq \mathtt{cur}(\mathtt{app} \circ (\mathtt{id}_{X^{Y'}} \times f)) : X^{Y'} \to X^Y$$

since it is the case that $\begin{cases} X^{\mathtt{id}_Y} & = \mathtt{id}_{X^Y} \\ X^{g \circ f} & = X^f \circ X^g \end{cases}$

(see Exercise Sheet 3, question 5).

Note that since a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ preserves domains, codomains, composition and identity morphisms

it sends commutative diagrams in $\mathbf{C}$ to commutative diagrams in $\mathbf{D}$

E.g.

$$
\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dd, "h"] \\
Y \arrow[dr, "g"'] & \\
& Z
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
& F\,X \arrow[dl, "F\,f"'] \arrow[dd, "F\,h = F(g \circ f) = F\,g \circ F\,f"] \\
F\,Y \arrow[dr, "F\,g"'] & \\
& F\,Z
\end{tikzcd}
$$

Note that since a functor $F : \mathbf{C} \to \mathbf{D}$ preserves domains, codomains, composition and identity morphisms

it sends isomorphisms in $\mathbf{C}$ to isomorphisms in $\mathbf{D}$, because

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathtt{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathtt{id}_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
F\,X \arrow[r, "F\,f"] \arrow[dr, "\mathtt{id}_{F\,X}"'] & F\,Y \arrow[d, "F\,g"] \arrow[dr, "\mathtt{id}_{F\,Y}"] & \\
& F\,X \arrow[r, "F\,f"'] & F\,Y
\end{tikzcd}
$$

so $\boxed{F(f^{-1}) = (F\,f)^{-1}}$

Composing functors

Given functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$, we get a functor $\boxed{G \circ F : \mathbf{C} \to \mathbf{E}}$ with

$$G \circ F \left(\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ Y \end{array} \right) = \begin{array}{c} G(F\,X) \\ \downarrow{\scriptstyle G(F\,f)} \\ G(F\,Y) \end{array}$$

(this preserves composition and identity morphisms, because F and G do)

Identity functor

on a category $\mathbf{C}$ is $\boxed{\mathtt{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}}$ where

$$\mathtt{id}_{\mathbf{C}} \left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \right) = \begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Functor composition and identity functors satisfy

associativity	$H \circ (G \circ F) = (H \circ G) \circ F$
unity	$\mathtt{id}_{\mathbf{D}} \circ F = F = F \circ \mathtt{id}_{\mathbf{C}}$

So we can get categories whose objects are categories and whose morphisms are functors

but we have to be a bit careful about size...

Size

One of the axioms of set theory is

set membership is a well-founded relation, that is, there is no infinite sequence of sets $X_0, X_1, X_2, \dots$ with

$$\cdots \in X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0$$

So in particular there is no set X with $X \in X$.

So we cannot form the "set of all sets" or the "category of all categories".

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So we cannot form the "set of all sets" or the "category of all categories".

But we do assume there are (lots of) big sets

$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$$

where "big" means each $\mathcal{U}_n$ is a Grothendieck universe…

Grothendieck universes

A Grothendieck universe $\mathscr{U}$ is a set of sets satisfying

- $X \in Y \in \mathscr{U} \Rightarrow X \in \mathscr{U}$
- $X, Y \in \mathscr{U} \Rightarrow \{X, Y\} \in \mathscr{U}$
- $X \in \mathscr{U} \Rightarrow \mathscr{P} X \triangleq \{Y \mid Y \subseteq X\} \in \mathscr{U}$
- $X \in U \wedge F \in \mathscr{U}^X \Rightarrow$
  $\{y \mid \exists x \in X,\ y \in F\, x\} \in \mathscr{U}$

  (hence also $X, Y \in \mathscr{U} \ \Rightarrow\ X \times Y \in \mathscr{U} \ \wedge\ Y^X \in \mathscr{U}$)

The above properties are satisfied by $\mathscr{U} = \emptyset$, but we will always assume

- $\mathbb{N} \in \mathscr{U}$

Size

We assume

there is an infinite sequence $\mathscr{U}_0 \in \mathscr{U}_1 \in \mathscr{U}_2 \in \cdots$ of bigger and bigger Grothendieck universes

and revise the previous definition of "the" category of sets and functions:

$\mathbf{Set}_n$ = category whose objects are all the sets in $\mathscr{U}_n$ and with $\mathbf{Set}_n(X, Y) = Y^X =$ all functions from X to Y.

Notation: $\boxed{\mathbf{Set} \triangleq \mathbf{Set}_0}$ — its objects are called small sets (and other sets we call large).

Size

$\mathbf{Set}$ is the category of small sets.

Definition. A category $\mathbf{C}$ is locally small if for all $X, Y \in \mathbf{C}$, the set of $\mathbf{C}$-morphisms $X \to Y$ is small, that is, $\mathbf{C}(X, Y) \in \mathbf{Set}$.

$\mathbf{C}$ is a small category if it is both locally small and $\mathtt{obj}\, \mathbf{C} \in \mathbf{Set}$.

E.g. $\mathbf{Set}$, $\mathbf{Preord}$ and $\mathbf{Mon}$ are all locally small (but not small).

Given $P \in \mathbf{Preord}$, the category $\mathbf{C}_P$ it determines is small; similarly, the category $\mathbf{C}_M$ determined by $M \in \mathbf{Mon}$ is small.

The category of small categories, $\mathbf{Cat}$

- objects are all small categories
- morphisms in $\mathbf{Cat}(\mathbf{C}, \mathbf{D})$ are all functors $\mathbf{C} \to \mathbf{D}$
- composition and identity morphisms as for functors

$\mathbf{Cat}$ is a locally small category