# Category Theory

## Lecture 9

# STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

Qu: which equations are always satisfied in any ccc?

Ans: $\beta\eta$-equivalence...

# STLC $\beta\eta$-Equality

The relation $\boxed{\Gamma \vdash s =_{\beta\eta} t : A}$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

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- $\beta$-conversions

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash s : A}{\Gamma \vdash (\lambda x : A.\, t)s =_{\beta\eta} t[s/x] : B}$$

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{fst}(s\,,\,t) =_{\beta\eta} s : A}$$

$$\frac{\Gamma \vdash s : A \qquad \Gamma \vdash t : B}{\Gamma \vdash \mathtt{snd}(s\,,\,t) =_{\beta\eta} t : B}$$

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- $\beta$-conversions
- $\eta$-conversions

$$\frac{\Gamma \vdash t : A \to B \qquad x \text{ does not occur in } t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A.\, t\, x) : A \to B}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t =_{\beta\eta} (\mathtt{fst}\, t\,,\, \mathtt{snd}\, t) : A \times B} \qquad \frac{\Gamma \vdash t : \mathtt{unit}}{\Gamma \vdash t =_{\beta\eta} () : \mathtt{unit}}$$

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- $\beta$-conversions
- $\eta$-conversions
- congruence rules

$$\frac{\Gamma, x : A \vdash t =_{\beta\eta} t' : B}{\Gamma \vdash \lambda x : A.\, t =_{\beta\eta} \lambda x : A.\, t' : A \to B}$$

$$\frac{\Gamma \vdash s =_{\beta\eta} s' : A \to B \qquad \Gamma \vdash t =_{\beta\eta} t' : A}{\Gamma \vdash s\, t =_{\beta\eta} s'\, t' : B}$$

etc

# STLC $\beta\eta$-Equality

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- $\beta$-conversions
- $\eta$-conversions
- congruence rules
- $=_{\beta\eta}$ is reflexive, symmetric and transitive

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t =_{\beta\eta} t : A} \qquad \frac{\Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash t =_{\beta\eta} s : A}$$

$$\frac{\Gamma \vdash r =_{\beta\eta} s : A \qquad \Gamma \vdash s =_{\beta\eta} t : A}{\Gamma \vdash r =_{\beta\eta} t : A}$$

# STLC $\beta\eta$-Equality

**Soundness Theorem** for semantics of STLC in a ccc.
If $\Gamma \vdash s =_{\beta\eta} t : A$ is provable, then in any ccc

$$M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$$

are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \to M[\![A]\!]$.

**Proof** is by induction on the structure of the proof of $\Gamma \vdash s =_{\beta\eta} t : A$.

Here we just check the case of $\beta$-conversion for functions.

So suppose we have $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

$$M[\![\Gamma \vdash (\lambda x : A.\, t)s : B]\!] = M[\![\Gamma \vdash t[s/x] : B]\!]$$

Suppose

$$M[\![\Gamma]\!] = X$$
$$M[\![A]\!] = Y$$
$$M[\![B]\!] = Z$$
$$M[\![\Gamma, x : A \vdash t : B]\!] = f : X \times Y \longrightarrow Z$$
$$M[\![\Gamma \vdash s : A]\!] = g : X \longrightarrow Z$$

Then

$$M[\![\Gamma \vdash \lambda x : A.\, t : A \to B]\!] = \mathtt{cur}\, f : X \longrightarrow Z^Y$$

and hence

$$
\begin{aligned}
& M[\![\Gamma \vdash (\lambda x : A.\, t)s : B]\!] \\
& = \mathtt{app} \circ \langle \mathtt{cur}\, f\,, g \rangle \\
& ? \\
& = f \circ \langle \mathtt{id}_X\,, g \rangle && \text{by definition of } \mathtt{cur}\, f \\
& = M[\![\Gamma \vdash t[s/x] : B]\!] && \text{by the Substitution Theorem}
\end{aligned}
$$

as required.

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$$M[\![\Gamma, x : A \vdash t : B]\!] = f : X \times Y \longrightarrow Z$$
$$M[\![\Gamma \vdash s : A]\!] = g : X \longrightarrow Z$$

Then

$$M[\![\Gamma \vdash \lambda x : A.\, t : A \to B]\!] = \mathtt{cur}\, f : X \longrightarrow Z^Y$$

and hence

$$M[\![\Gamma \vdash (\lambda x : A.\, t)s : B]\!]$$
$$= \mathtt{app} \circ \langle \mathtt{cur}\, f\,, g\rangle$$
$$= \mathtt{app} \circ (\mathtt{cur}\, f \times \mathtt{id}_Y) \circ \langle \mathtt{id}_X\,, g\rangle \qquad \text{since } (a \times b) \circ \langle c\,, d\rangle = \langle a \circ c\,, b \circ d\rangle$$
$$= f \circ \langle \mathtt{id}_X\,, g\rangle \qquad \text{by definition of } \mathtt{cur}\, f$$
$$= M[\![\Gamma \vdash t[s/x] : B]\!] \qquad \text{by the Substitution Theorem}$$

as required.

$$\frac{\Gamma \vdash t : A \to B \qquad x \text{ does not occur in } t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A.\, t\, x) : A \to B}$$

$$[\![ \lambda x : A.\, t\, x ]\!] = \mathrm{cur}\, [\![ x : A \vdash t\, x ]\!]$$

$$= \mathrm{cur}\big(\mathrm{app} \circ \langle [\![ x : A \vdash t ]\!], \pi_2 \rangle\big)$$

FACT :

## Weakening Lemma

If $\Gamma \vdash t : B$ holds and $x \notin \mathrm{dom}\,\Gamma$,
then $\Gamma, x : A \vdash t : B$ also holds.

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Furthermore

$$\begin{array}{ccc} [\![\Gamma, x:A]\!] & \xrightarrow{[\![\Gamma, x:A \vdash t:B]\!]} & [\![B]\!] \\ \| & & \uparrow{\scriptstyle [\![\Gamma \vdash t:B]\!]} \\ [\![\Gamma]\!] \times [\![A]\!] & \xrightarrow{\pi_1} & [\![\Gamma]\!] \end{array}$$

always commutes in any ccc.

$$\frac{\Gamma \vdash t : A \to B \qquad x \text{ does not occur in } t}{\Gamma \vdash t =_{\beta\eta} (\lambda x : A .\, t\, x) : A \to B}$$

$$\begin{aligned}
[\![ \lambda x : A .\, t\, x ]\!] &= cur\, [\![ x : A \vdash t\, x ]\!] \\
&= cur(app \circ \langle [\![ x : A \vdash t ]\!], \pi_2 \rangle) \\
&= cur(app \circ \langle [\![ t ]\!] \circ \pi_1, \pi_2 \rangle)
\end{aligned}$$

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$$\begin{aligned}
[\![ \lambda x : A.\, t\, x ]\!] &= \mathrm{cur}\, [\![ x : A \vdash t\, x ]\!] \\
&= \mathrm{cur}\big(\mathrm{app} \circ \langle [\![ x : A \vdash t ]\!], \pi_2 \rangle\big) \\
&= \mathrm{cur}\big(\mathrm{app} \circ \langle [\![ t ]\!] \circ \pi_1, \pi_2 \rangle\big) \\
&= \mathrm{cur}\big(\mathrm{app} \circ ([\![ t ]\!] \times \mathrm{id}) \circ \underbrace{\langle \pi_1, \pi_2 \rangle}_{= \mathrm{id}}\big)
\end{aligned}$$

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[\![ \lambda x : A .\, t\, x ]\!] &= \mathrm{cur}\, [\![ x : A \vdash t\, x ]\!] \\
&= \mathrm{cur}\big(\mathrm{app} \circ \langle [\![ x : A \vdash t ]\!], \pi_2 \rangle\big) \\
&= \mathrm{cur}\big(\mathrm{app} \circ \langle [\![ t ]\!] \circ \pi_1, \pi_2 \rangle\big) \\
&= \mathrm{cur}\big(\mathrm{app} \circ ([\![ t ]\!] \times \mathrm{id}) \circ \underbrace{\langle \pi_1, \pi_2 \rangle}_{= \mathrm{id}}\big) \\
&= [\![ t ]\!]
\end{aligned}$$

# The internal language of a ccc, $\mathbf{C}$

- one ground type for each $\mathbf{C}$-object $X$
- for each $X \in \mathbf{C}$, one constant $f^X$ for each $\mathbf{C}$-morphism $f : 1 \to X$ ("global element" of the object $X$)

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of $\mathbf{C}$ using its cartesian closed structure, but in an "element-theoretic" way.

For example…

# Example

In any ccc $\mathbf{C}$, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

$$Z^{(X \times Y)} \cong (Z^Y)^X$$

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In any ccc $\mathbf{C}$, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

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which in the internal language of $\mathbf{C}$ is described by the terms

$$\diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z))$$
$$\diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z)$$

where $\begin{cases} s & \triangleq \lambda f : (X \times Y) \to Z.\, \lambda x : X.\, \lambda y : Y.\, f\,(x\,,\,y) \\ t & \triangleq \lambda g : X \to (Y \to Z).\, \lambda z : X \times Y.\, g\,(\mathtt{fst}\,z)\,(\mathtt{snd}\,z) \end{cases}$ and

which satisfy $\begin{cases} \diamond, f : (X \times Y) \to Z \vdash t(s\,f) =_{\beta\eta} f \\ \diamond, g : X \to (Y \to Z) \vdash s(t\,g) =_{\beta\eta} g \end{cases}$

# Free cartesian closed categories

The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc $\boxed{\mathbf{F}}$ (the free ccc for that language) with an interpretation function $M$ so that $\Gamma \vdash s =_{\beta\eta} t : A$ is provable iff $M[\![\Gamma \vdash s : A]\!] = M[\![\Gamma \vdash t : A]\!]$ in $\mathbf{F}$.

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- $\mathbf{F}$-objects are the STLC types over the given set of ground types
- $\mathbf{F}$-morphisms $A \longrightarrow B$ are equivalence classes of STLC terms $t$ satisfying $\diamond \vdash t : A \to B$ (so $t$ is a *closed* term—it has no free variables) with respect to the equivalence relation equating $s$ and $t$ if $\diamond \vdash s =_{\beta\eta} t : A \to B$ is provable.
- identity morphism on $A$ is the equivalence class of $\diamond \vdash \lambda x : A.\, x : A \to A$.
- composition of a morphism $A \longrightarrow B$ represented by $\diamond \vdash s : A \to B$ and a morphism $B \longrightarrow C$ represented by $\diamond \vdash t : B \to C$ is represented by $\diamond \vdash \lambda x : A.\, t(s\, x) : A \to C$.

# Curry-Howard correspondence

| **Logic** | | **Type Theory** |
|---|---|---|
| propositions | $\leftrightarrow$ | types |
| proofs | $\leftrightarrow$ | terms |

E.g. IPL *versus* STLC.

# Curry-Howard for IPL *vs* STLC

Proof of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL

$$
\dfrac{
  \dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\Phi \vdash \psi \Rightarrow \theta}\,(\text{WK})
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Phi \vdash \varphi \Rightarrow \psi}\,(\text{WK})
    \qquad
    \dfrac{}{\Phi \vdash \varphi}\,(\text{AX})
  }{\Phi \vdash \psi}\,({\Rightarrow}\text{E})
}{
  \dfrac{\Phi \vdash \theta}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta}\,({\Rightarrow}\text{I})
}\,({\Rightarrow}\text{E})
$$

where $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

# Curry-Howard for IPL *vs* STLC

and a corresponding STLC term

$$
\dfrac{
  \dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\Phi \vdash z : \psi \Rightarrow \theta}\,(\text{WK})
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\cdots}\,(\text{AX})}{\cdots}\,(\text{WK})}{\Phi \vdash y : \varphi \Rightarrow \psi}\,(\text{WK})
    \qquad
    \dfrac{}{\Phi \vdash x : \varphi}\,(\text{AX})
  }{\Phi \vdash y\,x : \psi}\,(\Rightarrow\text{E})
}{
  \dfrac{\Phi \vdash z(y\,x) : \theta}{\diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta \vdash \lambda x : \varphi.\, z(y\,x) : \varphi \Rightarrow \theta}\,(\Rightarrow\text{I})
}\,(\Rightarrow\text{E})
$$

where $\Phi = \diamond, y : \varphi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \varphi$

# Curry-Howard-Lawvere/Lambek correspondence

| Logic | | Type Theory | | Category Theory |
|---|---|---|---|---|
| propositions | $\leftrightarrow$ | types | $\leftrightarrow$ | objects |
| proofs | $\leftrightarrow$ | terms | $\leftrightarrow$ | morphisms |

E.g. IPL *versus* STLC *versus* CCCs

# Curry-Howard-Lawvere/Lambek correspondence

| **Logic** | | **Type Theory** | | **Category Theory** |
|---|---|---|---|---|
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E.g. IPL *versus* STLC *versus* CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.