Category Theory

Lecture 8

Recall:

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \ldots ::=$

$G, G', G'' \ldots$	"ground" types
$\mathtt{unit}$	unit type
$A \times B$	product type
$A \to B$	function type

Terms: $s, t, r, \ldots ::=$

c^A	constants (of given type A)
x	variable (countably many)
$()$	unit value
$(s\,,\,t)$	pair
$\mathtt{fst}\,t \quad \mathtt{snd}\,t$	projections
$\lambda x : A.\,t$	function abstraction
$s\,t$	function application

Semantics of STLC terms in a ccc

Given a cartesian closed category $\mathbf{C}$,

given any function M mapping

- ground types G to $\mathbf{C}$-objects $M(G)$
  (which extends to a function mapping all types to objects, $A \mapsto M[\![A]\!]$, as we have seen)

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- ground types G to $\mathbf{C}$-objects $M(G)$
- constants c^A to $\mathbf{C}$-morphisms $M(c^A) : 1 \longrightarrow M[\![A]\!]$

  (In a category with a terminal object 1, given an object $X \in \mathbf{C}$, morphisms $1 \to X$ are sometimes called global elements of X.)

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we get a function mapping provable instances of the typing relation $\Gamma \vdash t : A$ to $\mathbf{C}$-morphisms

$$\boxed{M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \to M[\![A]\!]}$$

defined by recursing over the proof of $\Gamma \vdash t : A$ from the typing rules (which follows the structure of t):

Semantics of STLC terms in a ccc

Variables:

$$M[\![\Gamma, x : A \vdash x : A]\!] = M[\![\Gamma]\!] \times M[\![A]\!] \xrightarrow{\pi_2} M[\![A]\!]$$

$$M[\![\Gamma, x' : A' \vdash x : A]\!] =$$

$$M[\![\Gamma]\!] \times M[\![A']\!] \xrightarrow{\pi_1} M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash x : A]\!]} M[\![A]\!]$$

Constants:

$$M[\![\Gamma \vdash c^A : A]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1 \xrightarrow{M(c^A)} M[\![A]\!]$$

Unit value:

$$M[\![\Gamma \vdash () : \mathtt{unit}]\!] = M[\![\Gamma]\!] \xrightarrow{\langle\rangle} 1$$

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,\,t) : A \times B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s:A]\!], M[\![\Gamma \vdash t:B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \mathtt{fst}\, t : A]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t:A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_1} M[\![A]\!]$$

Semantics of STLC terms in a ccc

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Projections:

Given that $\Gamma \vdash \mathtt{fst}\, t : A$ holds, there is a unique type B such that $\Gamma \vdash t : A \times B$ already holds.

$$M[\![\Gamma \vdash \mathtt{fst}\, t : A]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t : A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_1} M[\![A]\!]$$

Lemma. If $\Gamma \vdash t : A$ and $\Gamma \vdash t : B$ are provable, then $A = B$.

Semantics of STLC terms in a ccc

Pairing:

$$M[\![\Gamma \vdash (s\,,\,t) : A \times B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle M[\![\Gamma \vdash s : A]\!], M[\![\Gamma \vdash t : B]\!]\rangle} M[\![A]\!] \times M[\![B]\!]$$

Projections:

$$M[\![\Gamma \vdash \mathtt{snd}\, t : B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{M[\![\Gamma \vdash t : A \times B]\!]} M[\![A]\!] \times M[\![B]\!] \xrightarrow{\pi_2} M[\![B]\!]$$

(As for the case of $\mathtt{fst}$, if $\Gamma \vdash \mathtt{snd}\, t : B$, then $\Gamma \vdash t : A \times B$ already holds for a unique type A.)

Semantics of STLC terms in a ccc

Function abstraction:

$$M[\![\Gamma \vdash \lambda x : A.t : A \to B]\!] = \\ \mathtt{cur}\, f : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \to M[\![B]\!])$$

where

$$f = M[\![\Gamma, x : A \vdash t : B]\!] : M[\![\Gamma]\!] \times M[\![A]\!] \longrightarrow M[\![B]\!]$$

Semantics of STLC terms in a ccc

Function application:

$$M[\![\Gamma \vdash s\, t : B]\!] =$$

$$M[\![\Gamma]\!] \xrightarrow{\langle f, g\rangle} (M[\![A]\!] \to M[\![B]\!]) \times M[\![A]\!] \xrightarrow{\mathtt{app}} M[\![B]\!]$$

where

A = unique type such that $\Gamma \vdash s : A \to B$ and $\Gamma \vdash t : A$ already holds (exists because $\Gamma \vdash s\, t : B$ holds)

$f = M[\![\Gamma \vdash s : A \to B]\!] : M[\![\Gamma]\!] \longrightarrow (M[\![A]\!] \to M[\![B]\!])$

$g = M[\![\Gamma \vdash t : A]\!] : M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$

Example

Consider $\boxed{t \triangleq \lambda x : A.\, g(f\, x)}$ so that $\Gamma \vdash t : A \to C$ when $\Gamma \triangleq \diamond, f : A \to B, g : B \to C$.

Suppose $M[\![A]\!] = X$, $M[\![B]\!] = Y$ and $M[\![C]\!] = Z$ in $\mathbf{C}$. Then

$$
\begin{aligned}
M[\![\Gamma]\!] &= (1 \times Y^X) \times Z^Y \\
M[\![\Gamma, x : A]\!] &= ((1 \times Y^X) \times Z^Y) \times X
\end{aligned}
$$

$$
\begin{aligned}
M[\![\Gamma, x : A \vdash x : A]\!] &= \pi_2 \\
M[\![\Gamma, x : A \vdash g : B \to C]\!] &= \pi_2 \circ \pi_1 \\
M[\![\Gamma, x : A \vdash f : A \to B]\!] &= \pi_2 \circ \pi_1 \circ \pi_1 \\
M[\![\Gamma, x : A \vdash f\, x : B]\!] &= \mathtt{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 , \pi_2 \rangle \\
M[\![\Gamma, x : A \vdash g\,(f\, x) : C]\!] &= \mathtt{app} \circ \langle \pi_2 \circ \pi_1 , \mathtt{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 , \pi_2 \rangle \rangle \\
M[\![\Gamma \vdash t : A \to C]\!] &= \mathtt{cur}(\mathtt{app} \circ \langle \pi_2 \circ \pi_1 , \mathtt{app} \circ \langle \pi_2 \circ \pi_1 \circ \pi_1 , \pi_2 \rangle \rangle)
\end{aligned}
$$

STLC equations

take the form $\boxed{\Gamma \vdash s = t : A}$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\![\Gamma \vdash s : A]\!]$ and $M[\![\Gamma \vdash t : A]\!]$ are equal $\mathbf{C}$-morphisms $M[\![\Gamma]\!] \longrightarrow M[\![A]\!]$.

Qu: which equations are always satisfied in any ccc?

STLC equations

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Qu: which equations are always satisfied in any ccc?

Ans: $(\alpha)\beta\eta$-equivalence — to define this, first have to define alpha-equivalence, substitution and its semantics.

Alpha equivalence of STLC terms

The names of λ-bound variables should not affect meaning.

E.g. $\lambda f : A \to B.\, \lambda x : A.\, f\, x$ should have the same meaning as $\lambda x : A \to B.\, \lambda y : A.\, x\, y$

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This issue is best dealt with at the level of syntax rather than semantics: from now on we re-define "STLC term" to mean not an abstract syntax tree (generated as described before), but rather an equivalence class of such trees with respect to alpha-equivalence $\boxed{s =_\alpha t}$, defined as follows...

(Alternatively, one can use a "nameless" (de Bruijn) representation of terms.)

Alpha equivalence of STLC terms

$$\frac{}{c^A =_\alpha c^A} \qquad \frac{}{x =_\alpha x} \qquad \frac{}{() =_\alpha ()} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{(s\,,t) =_\alpha (s'\,,t')} \qquad \frac{t =_\alpha t'}{\mathtt{fst}\,t =_\alpha \mathtt{fst}\,t'}$$

$$\frac{t =_\alpha t'}{\mathtt{snd}\,t =_\alpha \mathtt{snd}\,t'} \qquad \frac{s =_\alpha s' \qquad t =_\alpha t'}{s\,t =_\alpha s'\,t'}$$

$$\frac{(y\ x) \cdot t =_\alpha (y\ x') \cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

Alpha equivalence of STLC terms

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result of replacing all occurrences of x with y in t

Alpha equivalence of STLC terms

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$$\frac{(y\ x) \cdot t =_\alpha (y\ x') \cdot t' \qquad y \text{ does not occur in } \{x, x', t, t'\}}{\lambda x : A.\, t =_\alpha \lambda x' : A.\, t'}$$

E.g.

$\lambda x : A.\, x\, x =_\alpha \lambda y : A.\, y\, y \neq_\alpha \lambda x : A.\, x\, y$

$(\lambda y : A.\, y)\, x =_\alpha (\lambda x : A.\, x)\, x \neq_\alpha (\lambda x : A.\, x)\, y$

Substitution

$\boxed{t[s/x]}$ = result of replacing all free occurrences of variable x in term t (i.e. those not occurring within the scope of a $\lambda x : A.\_$ binder) by the term s, alpha-converting λ-bound variables in t to avoid them "capturing" any free variables of t.

E.g. $(\lambda y : A.\,(y\,,\,x))[y/x]$ is $\lambda z : A.\,(z\,,\,y)$ and is not $\lambda y : A.\,(y\,,\,y)$

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The relation $t[s/x] = t'$ can be inductively defined by the following rules…

Substitution

$$\frac{}{c^A[s/x] = c^A} \qquad \frac{}{x[s/x] = s} \qquad \frac{y \neq x}{y[s/x] = y} \qquad \frac{}{()[s/x] = ()}$$

$$\frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1\,,\,t_2)[s/x] = (t_1'\,,\,t_2')} \qquad \frac{t[s/x] = t'}{(\mathtt{fst}\, t)[s/x] = \mathtt{fst}\, t'}$$

$$\frac{t[s/x] = t'}{(\mathtt{snd}\, t)[s/x] = \mathtt{snd}\, t'} \qquad \frac{t_1[s/x] = t_1' \qquad t_2[s/x] = t_2'}{(t_1\, t_2)[s/x] = t_1' t_2'}$$

$$\frac{t[s/x] = t' \qquad y \neq x \text{ and } y \text{ does not occur in } s}{(\lambda y : A.\, t)[s/x] = \lambda y : A.\, t'}$$

Semantics of substitution in a ccc

Substitution Lemma If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then so is $\Gamma \vdash t[s/x] : B$.

Substitution Theorem If $\Gamma \vdash s : A$ and $\Gamma, x : A \vdash t : B$ are provable, then in any ccc the following diagram commutes:

$$\begin{tikzcd}
M[\![\Gamma]\!] \arrow[r, "{\langle \mathtt{id}, M[\![\Gamma \vdash s : A]\!] \rangle}"] \arrow[dr, swap, "{M[\![\Gamma \vdash t[s/x] : B]\!]}"] & M[\![\Gamma]\!] \times M[\![A]\!] \arrow[d, "{M[\![\Gamma, x : A \vdash t : B]\!]}"] \\
 & M[\![B]\!]
\end{tikzcd}$$