

Category Theory

Lecture 7

Exercise Sheet 3

Solution notes for Ex.Sh. 2

} available
now

Examples class Tues 25 Oct 4-5pm

Part II in LT2

ACS in SW02

IPL entailment $\Phi \vdash \varphi$

Recall the rules:

$\frac{}{\Phi, \varphi \vdash \varphi} \text{ (AX)}$	$\frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi} \text{ (WK)}$	$\frac{\Phi \vdash \varphi \quad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi} \text{ (CUT)}$
--	--	--

$\frac{}{\Phi \vdash \text{true}} \text{ (TRUE)}$	$\frac{\Phi \vdash \varphi \quad \Phi \vdash \psi}{\Phi \vdash \varphi \& \psi} \text{ (&I)}$	$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi} \text{ (}\Rightarrow\text{I)}$
---	---	--

$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \varphi} \text{ (&E}_1\text{)}$	$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \psi} \text{ (&E}_2\text{)}$	$\frac{\Phi \vdash \varphi \Rightarrow \psi \quad \Phi \vdash \varphi}{\Phi \vdash \psi} \text{ (}\Rightarrow\text{E)}$
---	--	---

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots (AX)}{\Phi, \varphi \vdash \psi \Rightarrow \theta} (WK) \quad \frac{\frac{\frac{\dots (AX)}{\Phi, \varphi \vdash \varphi \Rightarrow \psi} (WK) \quad \frac{\dots (AX)}{\Phi, \varphi \vdash \varphi} (AX)}{\Phi, \varphi \vdash \psi} (\Rightarrow E)}{\Phi, \varphi \vdash \theta} (\Rightarrow E)}{\Phi \vdash \varphi \Rightarrow \theta} (\Rightarrow I) \quad \text{where } \Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$$

$$\frac{\frac{\frac{\frac{\dots (AX)}{\Psi \vdash \varphi \Rightarrow \psi} (WK) \quad \frac{\dots (AX)}{\Psi \vdash \varphi} (AX)}{\Psi \vdash \psi} (\Rightarrow E) \quad \frac{\frac{\frac{\dots (AX)}{\Psi, \psi \vdash \psi \Rightarrow \theta} (WK) \quad \frac{\dots (AX)}{\Psi, \psi \vdash \psi} (AX)}{\Psi, \psi \vdash \theta} (\Rightarrow E)}{\Psi \vdash \theta} (CUT)}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta} (\Rightarrow I) \quad \text{where } \Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots}{\Phi, \varphi \vdash \psi \Rightarrow \theta} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\frac{\dots}{\Phi, \varphi \vdash \varphi \Rightarrow \psi} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\dots}{\Phi, \varphi \vdash \varphi} \text{(AX)}}{\dots} \text{(}\Rightarrow\text{E)}}{\Phi, \varphi \vdash \psi} \text{(}\Rightarrow\text{E)}}{\Phi, \varphi \vdash \theta} \text{(}\Rightarrow\text{E)}}{\Phi \vdash \varphi \Rightarrow \theta} \text{(}\Rightarrow\text{I)}$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots}{\Psi \vdash \varphi \Rightarrow \psi} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\dots}{\Psi \vdash \varphi} \text{(AX)}}{\dots} \text{(}\Rightarrow\text{E)}}{\Psi \vdash \psi} \quad \frac{\frac{\frac{\dots}{\Psi, \psi \vdash \psi \Rightarrow \theta} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\dots}{\Psi, \psi \vdash \psi} \text{(AX)}}{\dots} \text{(}\Rightarrow\text{E)}}{\Psi, \psi \vdash \theta} \text{(CUT)}}{\Psi \vdash \theta} \text{(}\Rightarrow\text{I)}}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta} \text{(}\Rightarrow\text{I)}$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Why is the first proof simpler than the second one?

Proof theory

$\frac{}{\Phi, \varphi \vdash \varphi} \text{ (AX)}$	$\frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi} \text{ (WK)}$	$\frac{\Phi \vdash \varphi \quad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi} \text{ (CUT)}$
$\frac{}{\Phi \vdash \text{true}} \text{ (TRUE)}$	$\frac{\Phi \vdash \varphi \quad \Phi \vdash \psi}{\Phi \vdash \varphi \& \psi} \text{ (\&I)}$	$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi} \text{ (\Rightarrow I)}$
$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \varphi} \text{ (\&E}_1\text{)}$	$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \psi} \text{ (\&E}_2\text{)}$	$\frac{\Phi \vdash \varphi \Rightarrow \psi \quad \Phi \vdash \varphi}{\Phi \vdash \psi} \text{ (\Rightarrow E)}$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (cut) rule.

Proof theory

$\frac{}{\Phi, \varphi \vdash \varphi} \text{ (AX)}$	$\frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi} \text{ (WK)}$	$\frac{\Phi \vdash \varphi \quad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi} \text{ (CUT)}$
$\frac{}{\Phi \vdash \text{true}} \text{ (TRUE)}$	$\frac{\Phi \vdash \varphi \quad \Phi \vdash \psi}{\Phi \vdash \varphi \ \& \ \psi} \text{ (\&I)}$	$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi} \text{ (\Rightarrow I)}$
$\frac{\Phi \vdash \varphi \ \& \ \psi}{\Phi \vdash \varphi} \text{ (\&E}_1\text{)}$	$\frac{\Phi \vdash \varphi \ \& \ \psi}{\Phi \vdash \psi} \text{ (\&E}_2\text{)}$	$\frac{\Phi \vdash \varphi \Rightarrow \psi \quad \Phi \vdash \varphi}{\Phi \vdash \psi} \text{ (\Rightarrow E)}$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (cut) rule.

Simply-Typed Lambda Calculus provides a language for describing proofs in IPL and their properties...

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots (AX)}{\Phi, \varphi \vdash \psi \Rightarrow \theta} (WK)}{\Phi, \varphi \vdash \psi} (\Rightarrow E) \quad \frac{\frac{\frac{\dots (AX)}{\Phi, \varphi \vdash \varphi \Rightarrow \psi} (WK)}{\Phi, \varphi \vdash \psi} (\Rightarrow E)}{\Phi, \varphi \vdash \theta} (\Rightarrow E)}{\Phi \vdash \varphi \Rightarrow \theta} (\Rightarrow I)$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots (AX)}{\Psi \vdash \varphi \Rightarrow \psi} (WK)}{\Psi \vdash \psi} (\Rightarrow E) \quad \frac{\frac{\frac{\dots (AX)}{\Psi, \psi \vdash \psi \Rightarrow \theta} (WK)}{\Psi, \psi \vdash \theta} (\Rightarrow E)}{\Psi \vdash \theta} (CUT)}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta} (\Rightarrow I)$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

$\rightarrow \diamond, f : \varphi \Rightarrow \Psi, g : \Psi \Rightarrow \theta \vdash \lambda x : \varphi. (\lambda y : \Psi. g y)(f x) : \varphi \Rightarrow \theta$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots (AX)}{\Phi, \varphi \vdash \psi \Rightarrow \theta} (WK) \quad \frac{\frac{\frac{\dots (AX)}{\Phi, \varphi \vdash \varphi \Rightarrow \psi} (WK) \quad \frac{\dots (AX)}{\Phi, \varphi \vdash \varphi} (AX)}{\Phi, \varphi \vdash \psi} (\Rightarrow E)}{\Phi, \varphi \vdash \psi} (\Rightarrow E)}{\Phi, \varphi \vdash \theta} (\Rightarrow E)}{\Phi \vdash \varphi \Rightarrow \theta} (\Rightarrow I)$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\frac{\frac{\frac{\frac{\dots (AX)}{\Psi \vdash \varphi \Rightarrow \psi} (WK) \quad \frac{\dots (AX)}{\Psi \vdash \varphi} (AX)}{\Psi \vdash \psi} (\Rightarrow E) \quad \frac{\frac{\frac{\dots (AX)}{\Psi, \psi \vdash \psi \Rightarrow \theta} (WK) \quad \frac{\dots (AX)}{\Psi, \psi \vdash \psi} (AX)}{\Psi, \psi \vdash \theta} (\Rightarrow E)}{\Psi, \psi \vdash \theta} (\Rightarrow E)}{\Psi \vdash \theta} (CUT)}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta} (\Rightarrow I)$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

$\rightarrow \diamond, f : \varphi \Rightarrow \psi, g : \psi \Rightarrow \theta \vdash \lambda x : \varphi. (\lambda y : \psi. g y)(f x) : \varphi \Rightarrow \theta$
 $\rightarrow \diamond, f : \varphi \Rightarrow \psi, g : \psi \Rightarrow \theta \vdash \lambda x : \varphi. g(f x) : \varphi \Rightarrow \theta$

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \dots ::=$

$G, G', G'' \dots$ “ground” types

unit unit type

$A \times B$ product type

$A \rightarrow B$ function type

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \dots ::=$

$G, G', G'' \dots$	“ground” types
unit	unit type
$A \times B$	product type
$A \rightarrow B$	function type

Terms: $s, t, r, \dots ::=$

c^A	constants (of given type A)
x	variable (countably many)
$()$	unit value
(s, t)	pair
$\text{fst } t \quad \text{snd } t$	projections
$\lambda x : A. t$	function abstraction
$s t$	function application

STLC

Some examples of terms:

- ▶ $\lambda z : (A \rightarrow B) \times (A \rightarrow C). \lambda x : A. ((\text{fst } z) x, (\text{snd } z) x)$
(has type $((A \rightarrow B) \times (A \rightarrow C)) \rightarrow (A \rightarrow (B \times C))$)
- ▶ $\lambda z : A \rightarrow (B \times C). (\lambda x : A. \text{fst}(z x), \lambda y : A. \text{snd}(z y))$
(has type $(A \rightarrow (B \times C)) \rightarrow ((A \rightarrow B) \times (A \rightarrow C))$)
- ▶ $\lambda z : A \rightarrow (B \times C). \lambda x : A. ((\text{fst } z) x, (\text{snd } z) x)$
(has no type)

STLC typing relation, $\Gamma \vdash t : A$

Γ ranges over **typing environments**

$$\Gamma ::= \diamond \mid \Gamma, x : A$$

(so typing environments are comma-separated snoc-lists of (variable,type)-pairs
— in fact only the lists whose variables are mutually distinct get used)

The typing relation $\Gamma \vdash t : A$ is inductively defined by the following rules, which make use of the following notation

Γ ok means: no variable occurs more than once in Γ

$\text{dom } \Gamma$ = finite set of variables occurring in Γ

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for variables

$$\frac{\Gamma \text{ ok} \quad x \notin \text{dom } \Gamma}{\Gamma, x : A \vdash x : A} \text{ (VAR)}$$

$$\frac{\Gamma \vdash x : A \quad x' \notin \text{dom } \Gamma}{\Gamma, x' : A' \vdash x : A} \text{ (VAR')}$$

Typing rules for constants and unit value

$$\frac{\Gamma \text{ ok}}{\Gamma \vdash c^A : A} \text{ (CONS)}$$

$$\frac{\Gamma \text{ ok}}{\Gamma \vdash () : \text{unit}} \text{ (UNIT)}$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for pairs and projections

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash (s, t) : A \times B} \text{ (PAIR)}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \text{fst } t : A} \text{ (FST)}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \text{snd } t : B} \text{ (SND)}$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for function abstraction & application

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A. t : A \rightarrow B} \text{ (FUN)}$$

$$\frac{\Gamma \vdash s : A \rightarrow B \quad \Gamma \vdash t : A}{\Gamma \vdash s t : B} \text{ (APP)}$$

STLC typing relation, $\Gamma \vdash t : A$

Example typing derivation:

$$\begin{array}{c}
 \frac{}{\Gamma \vdash g : B \rightarrow C} \text{(VAR)} \quad \frac{}{\diamond, f : A \rightarrow B \vdash f : A \rightarrow B} \text{(VAR)} \\
 \frac{}{\Gamma, x : A \vdash g : B \rightarrow C} \text{(VAR')} \quad \frac{}{\Gamma \vdash f : A \rightarrow B} \text{(VAR')} \quad \frac{}{\Gamma, x : A \vdash x : A} \text{(VAR)} \\
 \frac{}{\Gamma, x : A \vdash g : B \rightarrow C} \text{(VAR')} \quad \frac{}{\Gamma, x : A \vdash f : A \rightarrow B} \text{(VAR')} \quad \frac{}{\Gamma, x : A \vdash f x : B} \text{(APP)} \\
 \frac{}{\Gamma, x : A \vdash g(f x) : C} \text{(FUN)} \\
 \frac{}{\Gamma \vdash \lambda x : A. g(f x) : A \rightarrow C} \text{(FUN)} \\
 \frac{}{\diamond, f : A \rightarrow B \vdash \lambda g : B \rightarrow C. \lambda x : A. g(f x) : (B \rightarrow C) \rightarrow (A \rightarrow C)} \text{(FUN)} \\
 \frac{}{\diamond \vdash \lambda f : A \rightarrow B. \lambda g : B \rightarrow C. \lambda x : A. g(f x) : (A \rightarrow B) \rightarrow (B \rightarrow C) \rightarrow (A \rightarrow C)} \text{(FUN)}
 \end{array}$$

where $\Gamma \triangleq \diamond, f : A \rightarrow B, g : B \rightarrow C$

N.B. the STLC typing rules are “syntax-directed”, by the structure of terms t and then in the case of variables x , by the structure of typing environments Γ .

Semantics of STLC types in a ccc

Given a cartesian closed category \mathbf{C} ,

any function M mapping ground types G to objects $M(G) \in \mathbf{C}$

extends to function $A \mapsto M[A] \in \mathbf{C}$ and $\Gamma \mapsto M[\Gamma] \in \mathbf{C}$ from STLC types and typing environments to \mathbf{C} -objects, by recursion on the structure of A :

$$M[G] = M(G)$$

$$M[\text{unit}] = 1$$

terminal object in \mathbf{C}

$$M[A \times B] = M[A] \times M[B]$$

product in \mathbf{C}

$$M[A \rightarrow B] = M[A] \rightarrow M[B]$$

exponential in \mathbf{C}

$$M[\diamond] = 1$$

terminal object in \mathbf{C}

$$M[\Gamma, x : A] = M[\Gamma] \times M[A]$$

product in \mathbf{C}