Category Theory

Lecture 4

- Solution notes for Ex. Sheet 1 on Moodle
- Ex. Sheet 2 on web page & Moodle
- Use the Discussion Forum on Moodle if you have questions
  (or email ampl12 @ cam)
Binary products

In a category $C$, a product for objects $X, Y \in C$ is a diagram $X \leftarrow P \rightarrow Y$ with the universal property:

For all $X \leftarrow Z \rightarrow Y$ in $C$, there is a unique $C$-morphism $h : Z \rightarrow P$ such that the following diagram commutes in $C$:

$$
\begin{array}{c}
\begin{array}{ccc}
Z & \downarrow h & Y \\
\downarrow f & & \downarrow g \\
X & \leftarrow P & \rightarrow Y
\end{array}
\end{array}
$$
Binary products

In a category $\mathbf{C}$, a **product** for objects $X, Y \in \mathbf{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ with the universal property:

For all $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbf{C}$, there is a unique $\mathbf{C}$-morphism $h : Z \rightarrow P$ such that $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$

So $(P, \pi_1, \pi_2)$ is a terminal object in the category with

- objects: $(Z, f, g)$ where $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathbf{C}$
- morphisms $h : (Z_1, f_1, g_1) \rightarrow (Z_2, f_2, g_2)$ are $h \in \mathbf{C}(Z_1, Z_2)$ such that $f_1 = f_2 \circ h$ and $g_1 = g_2 \circ h$
- composition and identities as in $\mathbf{C}$

So if it exists, the binary product of two objects in a category is unique up to (unique) isomorphism.
Binary products

In a category $\mathcal{C}$, a product for objects $X, Y \in \mathcal{C}$ is a diagram $X \xleftarrow{\pi_1} P \xrightarrow{\pi_2} Y$ with the universal property:

For all $X \xleftarrow{f} Z \xrightarrow{g} Y$ in $\mathcal{C}$, there is a unique $\mathcal{C}$-morphism $h : Z \to P$ such that $f = \pi_1 \circ h$ and $g = \pi_2 \circ h$

**N.B.** products of objects in a category do not always exist. For example in the category

\[
\begin{array}{ccc}
\text{id}_0 & 0 & \text{id}_1 \\
\circlearrowleft & & \circlearrowright \\
\end{array}
\]

two objects, no non-identity morphisms

the objects 0 and 1 do not have a product, because there is no diagram of the form $0 \xleftarrow{?} 1 \to 1$ in this category.
Notation for binary products

Assuming $\mathcal{C}$ has binary products of objects, the product of $X, Y \in \mathcal{C}$ is written

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

and given $X \xleftarrow{f} Z \xrightarrow{g} Y$, the unique $h : Z \to X \times Y$ with $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$ is written

$$\langle f, g \rangle : Z \to X \times Y$$
Examples of products

In \textbf{Set}, category-theoretic products are given by the usual cartesian product of sets (set of all ordered pairs)

\[ X \times Y = \{ (x, y) \mid x \in X \land y \in Y \} \]
\[ \pi_1(x, y) = x \]
\[ \pi_2(x, y) = y \]

because...
Examples of products

In Preord, can take product of \((P_1, \sqsubseteq_1)\) and \((P_2, \sqsubseteq_2)\) to be

\[(P_1 \times P_2, \sqsubseteq)\]

(product in Set)

\[(x_1, x_2) \sqsubseteq (y_1, y_2) \iff x_1 \sqsubseteq_1 y_1 \land x_2 \sqsubseteq_2 y_2\]
Examples of products

In Preord, can take product of \((P_1, \sqsubseteq_1)\) and \((P_2, \sqsubseteq_2)\) to be

\[(P_1 \times P_2, \sqsubseteq)\]

The projection functions \(P_1 \xleftarrow{\pi_1} P_1 \times P_2 \xrightarrow{\pi_2} P_2\) are monotone for this pre-order on \(P_1 \times P_2\) and have the universal property needed for a product in Preord (check).
Examples of products

In \textbf{Mon}, can take product of \((M_1, \cdot_1, e_1)\) and \((M_2, \cdot_2, e_2)\) to be

\[
(M_1 \times M_2, \cdot, (e_1, e_2))
\]

\[
(x_1, x_2) \cdot (y_1, y_2) = (x_1 \cdot_1 y_1, x_2 \cdot_2 y_2)
\]

product in \textbf{Set}
Examples of products

In \( \text{Mon} \), can take product of \((M_1, \cdot_1, e_1)\) and \((M_2, \cdot_2, e_2)\) to be

\[
(M_1 \times M_2, \cdot, (e_1, e_2))
\]

The projection functions \( M_1 \leftarrow M_1 \times M_2 \rightarrow M_2 \) are monoid morphisms for this monoid structure on \( M_1 \times M_2 \) and have the universal property needed for a product in \( \text{Mon} \) (check).
Examples of products

Recall that each pre-ordered set \((P, \sqsubseteq)\) determines a category \(C_P\).

Given \(p, q \in P = \text{obj } C_P\), the product \(p \times q\) (if it exists) is a greatest lower bound (or \(\text{glb}\), or \text{meet}\) for \(p\) and \(q\) in \((P, \sqsubseteq)\):

**lower bound:**
\[
p \times q \sqsubseteq p \land p \times q \sqsubseteq q
\]

**greatest** among all lower bounds:
\[
\forall r \in P, \ r \sqsubseteq p \land r \sqsubseteq q \implies r \sqsubseteq p \times q
\]

**Notation:** glbs are often written \(p \land q\) or \(p \sqcap q\)
Duality

A binary coproduct of two objects in a category $\mathcal{C}$ is their product in the category $\mathcal{C}^{\text{op}}$. 
Duality

A binary coproduct of two objects in a category $\mathcal{C}$ is their product in the category $\mathcal{C}^{\text{op}}$.

Thus the coproduct of $X, Y \in \mathcal{C}$ if it exists, is a diagram $X \xrightarrow{\text{inl}} X + Y \xleftarrow{\text{inr}} Y$ with the universal property:

$\forall (X \xrightarrow{f} Z \xleftarrow{g} Y), \exists! (X + Y \xrightarrow{h} Z), f = h \circ \text{inl} \land g = h \circ \text{inr}$
Duality

A binary coproduct of two objects in a category $C$ is their product in the category $C^{\text{op}}$.

E.g. in $\text{Set}$, the coproduct of $X$ and $Y$

$$
\begin{align*}
X & \xrightarrow{\text{inl}} X + Y \xleftarrow{\text{inr}} Y \\
\text{inl}(x) &= (0, x) \\
\text{inr}(y) &= (1, y)
\end{align*}
$$

is given by their disjoint union (tagged sum)

$$
X + Y = \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}
$$

(prove this)