

Category Theory

Exercise Sheet 4 [GRADED]

Please attempt all five questions.

Solutions to be submitted via Moodle

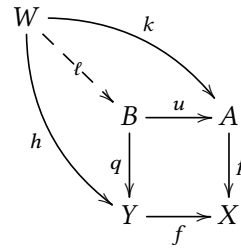
by 12:00 on Friday 4 November 2022

1. [10/40 marks] A *pullback square* in a category \mathbf{C} is a commutative diagram of the form

$$\begin{array}{ccc}
 B & \xrightarrow{u} & A \\
 q \downarrow & & \downarrow p \\
 Y & \xrightarrow{f} & X
 \end{array}
 \quad p \circ u = f \circ q
 \tag{1}$$

with the following universal property:

for all \mathbf{C} -objects W and \mathbf{C} -morphisms $Y \xleftarrow{h} W \xrightarrow{k} A$ satisfying $f \circ h = p \circ k$, there is a unique \mathbf{C} -morphism $\ell : W \rightarrow B$ satisfying $q \circ \ell = h$ and $u \circ \ell = k$



- (a) Let \mathbf{C} be a category and $f : Y \rightarrow X$ a morphism in \mathbf{C} . Show that f is a monomorphism (see Exercise Sheet 1, question 4) if and only if

$$\begin{array}{ccc}
 Y & \xrightarrow{\text{id}_Y} & Y \\
 \text{id}_Y \downarrow & & \downarrow f \\
 Y & \xrightarrow{f} & X
 \end{array}
 \tag{2}$$

is a pullback square in \mathbf{C} .

- (b) If (1) is a pullback square and p is a monomorphism, show that q is a monomorphism.
- (c) If (1) is a pullback square and p is an isomorphism, show that q is an isomorphism.
- (d) Given an example of a pullback square (1) in the category **Set** of sets and functions, for which q is an isomorphism, but p is not a monomorphism. (Recall that in **Set**, monomorphisms and isomorphisms are given by the functions that are respectively injective and bijective.)

2. [5/40 marks]

- (a) Given morphisms $X' \xrightarrow{f} X$ and $Y \xrightarrow{g} Y'$ in a cartesian closed category \mathbf{C} , show how to define a morphism $Y^X \rightarrow (Y')^{X'}$ in \mathbf{C} .

- (b) Given types A', A, B and B' in simply typed lambda calculus (STLC), give a term t satisfying

$$\diamond \vdash t : (A' \rightarrow A) \rightarrow (B \rightarrow B') \rightarrow (A \rightarrow B) \rightarrow (A' \rightarrow B')$$

If the semantics in a cartesian closed category of A', A, B and B' are the objects X', X, Y and Y' respectively, what is the semantics of t ?

3. [5/40 marks] Let $\mathbf{C} = \mathbf{Set}^{\text{op}}$ be the opposite category of the category \mathbf{Set} of sets and functions.

- (a) State, without proof, what is the product in \mathbf{C} of two objects X and Y .
 (b) Show by example that there are objects X and Y in \mathbf{C} for which there is no exponential and hence that \mathbf{C} is not a cartesian closed category.

4. [10/40 marks]

[In this question I use the notation $X \xrightarrow{\text{inl}_{X,Y}} X + Y \xleftarrow{\text{inr}_{X,Y}} Y$ for the coproduct (Lecture 4) of two object X and Y in a category, since it will be clearer to make explicit the objects X and Y in the notation for the associated coproduct injections, $\text{inl}_{X,Y}$ and $\text{inr}_{X,Y}$.]

A category \mathbf{C} is *distributive* if it has all binary products and binary coproducts, and for all objects $X, Y, Z \in \mathbf{C}$, (using the defining property of the coproduct $X \times Y \xrightarrow{\text{inl}_{X \times Y, X \times Z}} (X \times Y) + (X \times Z) \xleftarrow{\text{inr}_{X \times Y, X \times Z}} X \times Z$), the unique morphism $\delta_{X,Y,Z} : (X \times Y) + (X \times Z) \rightarrow X \times (Y + Z)$ that makes the following diagram commute

$$\begin{array}{ccc}
 X \times Y & & \\
 \text{inl}_{X \times Y, X \times Z} \downarrow & \searrow \text{id} \times \text{inl}_{Y,Z} & \\
 (X \times Y) + (X \times Z) & \xrightarrow{\delta_{X,Y,Z}} & X \times (Y + Z) \\
 \text{inr}_{X \times Y, X \times Z} \uparrow & \nearrow \text{id} \times \text{inr}_{Y,Z} & \\
 X \times Z & &
 \end{array} \tag{3}$$

is an isomorphism.

- (a) Using the usual product and coproduct constructs in the category \mathbf{Set} of sets and functions, show that it is a distributive category.
 (b) Give, with justification, an example of a category with binary products and coproducts that is not distributive.
 (c) If \mathbf{C} is a distributive category and 0 is an initial object in \mathbf{C} , prove that for all $X \in \mathbf{C}$, the unique morphism $0 \rightarrow X \times 0$ is an isomorphism.
5. [10/40 marks] A category \mathbf{C} is called *locally finite* if for all $X, Y \in \text{obj } \mathbf{C}$, the set of morphisms $\mathbf{C}(X, Y)$ is finite. \mathbf{C} is said to be *finite* if it is both locally finite and $\text{obj } \mathbf{C}$ is finite.

- (a) Prove that any finite category with binary products is a pre-order, that is, there is at most one morphism between any pair of objects. [Hint: if $f, g : X \rightarrow Y$ were distinct, use them to construct too large a number of morphisms from X to the product Y^n of Y with itself $n (> 0)$ times, for some suitable some number n .]
 (b) Is every locally finite category with binary products a pre-order? (Either prove it, or give a counterexample.)