1. Show that for any objects \( X \) and \( Y \) in a cartesian closed category \( C \), there are functions

\[
5 \colon C(X, Y) \to C(1, Y^X)
\]
\[
g \colon C(1, Y^X) \to \mathcal{C}(X, Y)
\]

that give a bijection between the set \( C(X, Y) \) of \( C \)-morphisms from \( X \) to \( Y \) and the set \( C(1, Y^X) \) of \( C \)-morphisms from the terminal object 1 to the exponential \( Y^X \). [Hint: use the isomorphism (7) from Exercise Sheet 2, question 2.]

2. Show that for any objects \( X \) and \( Y \) in a cartesian closed category \( C \), the morphism \( \text{app} : Y^X \times X \to Y \) satisfies \( \text{cur} \circ \text{app} = \text{id}_{Y^X} \). [Hint: recall from equation (4) on Exercise Sheet 2 that \( \text{id}_{Y^X} \times \text{id}_X = \text{id}_{Y^X \times X} \).]

3. Suppose \( f : Y \times X \to Z \) and \( g : W \to Y \) are morphisms in a cartesian closed category \( C \). Prove that

\[
\text{cur}(f \circ (g \times \text{id}_X)) = (\text{cur} f) \circ g \in C(W, Z^X)
\]

[Hint: use Exercise Sheet 2, question 1c.]

4. Let \( C \) be a cartesian closed category. For each \( C \)-object \( X \) and \( C \)-morphism \( f : Y \to Z \), define

\[
f^X \triangleq \text{cur}(Y^X \times X \xrightarrow{\text{app}} Y \xrightarrow{f} Z) \in C(Y^X, Z^X)
\]

(a) Prove that \( (\text{id}_Y)^X = \text{id}_{Y^X} \).

(b) Given \( f \in C(Y \times X, Z) \) and \( g \in C(Z, W) \), prove that

\[
\text{cur}(g \circ f) = g^X \circ \text{cur} f \in C(Y, W^X)
\]

(c) Deduce that if \( u \in C(Y, Z) \) and \( v \in C(Z, W) \), then \( (v \circ u)^X = v^X \circ u^X \in C(Y^X, W^X) \).

[Hint: for part (4a) use question 2; for part (4b) use Exercise Sheet 2, question 1c.]

5. Let \( C \) be a cartesian closed category. For each \( C \)-object \( X \) and \( C \)-morphism \( f : Y \to Z \), define

\[
X^f \triangleq \text{cur}(X^Z \times Y \xrightarrow{\text{app}} X^Z \times X \xrightarrow{\text{id}_X f} X) \in C(X^Z, X^Y)
\]

(a) Prove that \( X^{\text{id}_Y} = \text{id}_{X^Y} \).

(b) Given \( g \in C(W, X) \) and \( f \in C(Y \times X, Z) \), prove that

\[
\text{cur}(f \circ (\text{id}_Y \times g)) = Z^g \circ \text{cur} f \in C(Y, Z^W)
\]

(c) Deduce that if \( u \in C(Y, Z) \) and \( v \in C(Z, W) \), then \( X^{(v \circ u)} = X^v \circ X^u \in C(X^W, X^Y) \).

[Hint: for part (5a) use question 2; for part (5b) use Exercise Sheet 2, question 1c.]
6. Let $C$ be a cartesian closed category in which every pair of objects $X$ and $Y$ possesses a binary coproduct $X \overset{\text{inl}_{XY}}{\rightarrow} X + Y \overset{\text{inr}_{XY}}{\leftarrow} Y$. For all objects $X, Y, Z \in C$ construct an isomorphism $(Y + Z) \times X \cong (Y \times X) + (Z \times X)$. [Hint: you may find it helpful to use some of the properties from question 4.]

7. Using the natural deduction rules for Intuitionistic Propositional Logic (given in Lecture 6), give proofs of the following judgements. In each case write down a corresponding typing judgement of the Simply Typed Lambda Calculus.

(a) $\varnothing, \psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$
(b) $\varnothing, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$
(c) $\varnothing, ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi \vdash \varphi \Rightarrow \psi$

8. (a) Given simple types $A, B, C$, give terms $s$ and $t$ of the Simply Typed Lambda Calculus that satisfy the following typing and $\beta\eta$-equality judgements:

\[
\begin{align*}
\varnothing, x : (A \times B) \rightarrow C &\vdash s : A \rightarrow (B \rightarrow C) \quad (6) \\
\varnothing, y : A \rightarrow (B \rightarrow C) &\vdash t : (A \times B) \rightarrow C \quad (7) \\
\varnothing, x : (A \times B) \rightarrow C &\vdash t[s/y] =_{\beta\eta} x : (A \times B) \rightarrow C \quad (8) \\
\varnothing, y : A \rightarrow (B \rightarrow C) &\vdash s[t/x] =_{\beta\eta} y : A \rightarrow (B \rightarrow C) \quad (9)
\end{align*}
\]

(b) Explain why question (8a) implies that for any three objects $X, Y$ and $Z$ in a cartesian closed category $C$, there are morphisms

\[
\begin{align*}
f : Z^{(X \times Y)} &\rightarrow (Z^Y)^X \quad (10) \\
g : (Z^Y)^X &\rightarrow Z^{(X \times Y)} \quad (11)
\end{align*}
\]

that give an isomorphism $Z^{(X \times Y)} \cong (Z^Y)^X$ in $C$.

9. Make up and solve a question like question 8 ending with an isomorphism $X^1 \cong X$ for any object $X$ in a cartesian closed category $C$ (with terminal object 1).