University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 3

1. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, there are functions

$$f \in \mathbf{C}(X, Y) \mapsto \ulcorner f \urcorner \in \mathbf{C}(1, Y^X)$$
$$g \in \mathbf{C}(1, Y^X) \mapsto \overline{g} \in \mathbf{C}(X, Y)$$

that give a bijection between the set $\mathbf{C}(X, Y)$ of $\mathbf{C}$-morphisms from X to Y and the set $\mathbf{C}(1, Y^X)$ of $\mathbf{C}$-morphisms from the terminal object 1 to the exponential Y^X. [Hint: use the isomorphism (7) from Exercise Sheet 2, question 2.]

2. Show that for any objects X and Y in a cartesian closed category $\mathbf{C}$, the morphism $\mathtt{app} : Y^X \times X \to Y$ satisfies $\mathtt{cur}(\mathtt{app}) = \mathtt{id}_{Y^X}$. [Hint: recall from equation (4) on Exercise Sheet 2 that $\mathtt{id}_{Y^X} \times \mathtt{id}_X = \mathtt{id}_{Y^X \times X}$.]

3. Suppose $f : Y \times X \to Z$ and $g : W \to Y$ are morphisms in a cartesian closed category $\mathbf{C}$. Prove that

$$\mathtt{cur}(f \circ (g \times \mathtt{id}_X)) = (\mathtt{cur}\, f) \circ g \in \mathbf{C}(W, Z^X) \tag{1}$$

[Hint: use Exercise Sheet 2, question 1c.]

4. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define

$$f^X \triangleq \mathtt{cur}(Y^X \times X \xrightarrow{\mathtt{app}} Y \xrightarrow{f} Z) \in \mathbf{C}(Y^X, Z^X) \tag{2}$$

   (a) Prove that $(\mathtt{id}_Y)^X = \mathtt{id}_{Y^X}$.

   (b) Given $f \in \mathbf{C}(Y \times X, Z)$ and $g \in \mathbf{C}(Z, W)$, prove that

$$\mathtt{cur}(g \circ f) = g^X \circ \mathtt{cur}\, f \in \mathbf{C}(Y, W^X) \tag{3}$$

   (c) Deduce that if $u \in \mathbf{C}(Y, Z)$ and $v \in \mathbf{C}(Z, W)$, then $(v \circ u)^X = v^X \circ u^X \in \mathbf{C}(Y^X, W^X)$.

   [Hint: for part (4a) use question 2; for part (4b) use Exercise Sheet 2, question 1c.]

5. Let $\mathbf{C}$ be a cartesian closed category. For each $\mathbf{C}$-object X and $\mathbf{C}$-morphism $f : Y \to Z$, define

$$X^f \triangleq \mathtt{cur}(X^Z \times Y \xrightarrow{\mathtt{id} \times f} X^Z \times Z \xrightarrow{\mathtt{app}} X) \in \mathbf{C}(X^Z, X^Y) \tag{4}$$

   (a) Prove that $X^{\mathtt{id}_Y} = \mathtt{id}_{X^Y}$.

   (b) Given $g \in \mathbf{C}(W, X)$ and $f \in \mathbf{C}(Y \times X, Z)$, prove that

$$\mathtt{cur}(f \circ (\mathtt{id}_Y \times g)) = Z^g \circ \mathtt{cur}\, f \in \mathbf{C}(Y, Z^W) \tag{5}$$

   (c) Deduce that if $u \in \mathbf{C}(Y, Z)$ and $v \in \mathbf{C}(Z, W)$, then $X^{(v \circ u)} = X^u \circ X^v \in \mathbf{C}(X^W, X^Y)$.

   [Hint: for part (5a) use question 2; for part (5b) use Exercise Sheet 2, question 1c.]

6. Let $\mathbf{C}$ be a cartesian closed category in which every pair of objects X and Y possesses a binary coproduct $X \xrightarrow{\mathtt{inl}_{X,Y}} X + Y \xleftarrow{\mathtt{inr}_{X,Y}} Y$. For all objects $X, Y, Z \in \mathbf{C}$ construct an isomorphism $(Y + Z) \times X \cong (Y \times X) + (Z \times X)$. [Hint: you may find it helpful to use some of the properties from question 4.]

7. Using the natural deduction rules for Intuitionistic Propositional Logic (given in Lecture 6), give proofs of the following judgements. In each case write down a corresponding typing judgement of the Simply Typed Lambda Calculus.

   (a) $\diamond, \psi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$

   (b) $\diamond, \varphi \vdash (\varphi \Rightarrow \psi) \Rightarrow \psi$

   (c) $\diamond, ((\varphi \Rightarrow \psi) \Rightarrow \psi) \Rightarrow \psi \vdash \varphi \Rightarrow \psi$

8. (a) Given simple types A, B, C, give terms s and t of the Simply Typed Lambda Calculus that satisfy the following typing and $\beta\eta$-equality judgements:

$$\diamond, x : (A \times B) \rightarrow C \vdash s : A \rightarrow (B \rightarrow C) \tag{6}$$

$$\diamond, y : A \rightarrow (B \rightarrow C) \vdash t : (A \times B) \rightarrow C \tag{7}$$

$$\diamond, x : (A \times B) \rightarrow C \vdash t[s/y] =_{\beta\eta} x : (A \times B) \rightarrow C \tag{8}$$

$$\diamond, y : A \rightarrow (B \rightarrow C) \vdash s[t/x] =_{\beta\eta} y : A \rightarrow (B \rightarrow C) \tag{9}$$

   (b) Explain why question (8a) implies that for any three objects X,Y and Z in a cartesian closed category $\mathbf{C}$, there are morphisms

$$f : Z^{(X \times Y)} \longrightarrow (Z^Y)^X \tag{10}$$

$$g : (Z^Y)^X \longrightarrow Z^{(X \times Y)} \tag{11}$$

   that give an isomorphism $Z^{(X \times Y)} \cong (Z^Y)^X$ in $\mathbf{C}$.

9. Make up and solve a question like question 8 ending with an isomorphism $X^1 \cong X$ for any object X in a cartesian closed category $\mathbf{C}$ (with terminal object 1).