University of Cambridge
2022/23 Part II / Part III / MPhil ACS
Category Theory
Exercise Sheet 2

1. Let $C$ be a category with binary products.
   
   (a) For morphisms $f \in C(X, Y)$, $g_1 \in C(Y, Z_1)$ and $g_2 \in C(Y, Z_2)$, show that
   
   $$ (g_1, g_2) \circ f = (g_1 \circ f, g_2 \circ f) \in C(X, Z_1 \times Z_2) \quad (1) $$
   
   (b) For morphisms $f_1 \in C(X_1, Y_1)$ and $f_2 \in C(X_2, Y_2)$, define
   
   $$ f_1 \times f_2 \triangleq (f_1 \circ \pi_1, f_2 \circ \pi_2) \in C(X_1 \times X_2, Y_1 \times Y_2) \quad (2) $$
   
   For any $g_1 \in C(Z, X_1)$ and $g_2 \in C(Z, X_2)$, show that
   
   $$ (f_1 \times f_2) \circ (g_1, g_2) = (f_1 \circ g_1, f_2 \circ g_2) \in C(Z, Y_1 \times Y_2) \quad (3) $$
   
   (c) Show that the operation $f_1, f_2 : f_1 \times f_2$ defined in part (1b) satisfies
   
   $$ (h_1 \times h_2) \circ (k_1 \times k_2) = (h_1 \circ k_1) \times (h_2 \circ k_2) \quad (4) $$
   
   $$ \text{id}_X \times \text{id}_Y = \text{id}_{X \times Y} \quad (5) $$

2. Let $C$ be a category with binary products $\times$, and a terminal object $1$. Given objects $X, Y, Z \in C$, construct isomorphisms
   
   $$ \alpha_{X,Y,Z} : X \times (Y \times Z) \cong (X \times Y) \times Z \quad (6) $$
   
   $$ \lambda_X : 1 \times X \cong X \quad (7) $$
   
   $$ \rho_X : X \times 1 \cong X \quad (8) $$
   
   $$ \tau_{X,Y} : X \times Y \cong Y \times X \quad (9) $$

3. A pairing for a monoid $(M, \cdot, e)$ consists of elements $p_1, p_2 \in M$ and a binary operation $\langle \cdot, \cdot \rangle : M \times M \to M$ satisfying for all $x, y, z \in M$
   
   $$ p_1 \cdot \langle x, y \rangle = x \quad (10) $$
   
   $$ p_2 \cdot \langle x, y \rangle = y \quad (11) $$
   
   $$ \langle p_1, p_2 \rangle = e \quad (12) $$
   
   $$ \langle x, y \rangle \cdot z = \langle x \cdot z, y \cdot z \rangle \quad (13) $$

   Given such a pairing, show that the monoid, when regarded as a one-object category, has binary products.

4. A monoid $(M, \cdot_M, e_M)$ is said to be abelian if its multiplication is commutative: $(\forall x, y \in M) \ x \cdot_M y = y \cdot_M x$. 


(a) If \((M, \cdot_M, e_M)\) is an abelian monoid, show that the functions \(m \in \text{Set}(M \times M, M)\) and \(u \in \text{Set}(1, M)\) defined by

\[
m(x, y) = x \cdot_M y \quad \text{(all } x, y \in M\text{)}
\]

\[
u(0) = e_M
\]

determine morphisms in the category \(\text{Mon}\) of monoids, \(m \in \text{Mon}(M \times M, M)\) and \(u \in \text{Mon}(1, M)\) (where as usual we just write \(M\) for the monoid \((M, \cdot_M, e_M)\) and \(1\) for the terminal monoid \((1, \cdot_1, e_1)\) with \(1\) a one-element set, \(\{0\}\) say, \(0 \cdot_1 0 = 0\) and \(e_1 = 0\)).

Show further that \(m\) and \(u\) make the monoid \(M\) into a “monoid object in the category \(\text{Mon}\)”, in the sense that the following diagrams in \(\text{Mon}\) commute:

\[
\begin{align*}
\xymatrix{ \left( M \times M \right) \times M \ar[r]^{m \times \text{id}} & M \times M \ar[r]^{m} & M \\
\langle \pi_1 \circ \pi_2, \langle \pi_1 \circ \pi_2 \rangle \rangle \ar[r] \ar[r] & \pi_2 \circ \pi_2 \ar[r] & \text{id} \quad \text{(associativity)}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
1 \times M \ar[r]^{u \times \text{id}} & M \times M \ar[r]^{m} & M \\
\pi_2 \ar[r] & \text{id} \quad \text{(left unit)}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
M \ar[r]^{\text{id} \times u} & M \times M \ar[r]^{m} & M \\
\pi_1 \ar[r] & \text{id} \quad \text{(right unit)}
\end{array}
\end{align*}
\]

(b) Show that every monoid object in the category \(\text{Mon}\) (in the above sense) arises as in (4a).

[Hint: if necessary, search the internet for "Eckmann-Hilton argument".]

5. Let \(\text{AbMon}\) be the category whose objects are abelian monoids (question 4) and whose morphisms, identity morphisms and composition are as in \(\text{Mon}\).

(a) Show that the product in \(\text{Mon}\) of two abelian monoids gives their product in \(\text{AbMon}\).

(b) Given \(M, N \in \text{AbMon}\) define morphisms \(i \in \text{AbMon}(M \times N)\) and \(j \in \text{AbMon}(N, M \times N)\) that make \(M \times N\) into a coproduct in \(\text{AbMon}\).

6. The category \(\text{Set}^\omega\) of ’sets evolving through discrete time’ is defined as follows:

- Objects are triples \((X, (\_)^+, [\_])\), where \(X \in \text{Set}\), 
  \((\_)^+ \in \text{Set}(X, X)\) and 
  \([\_] \in \text{Set}(X, \mathbb{N})\)
  satisfy for all \(x \in X\)

\[
|x^+| = |x| + 1 \quad \text{(17)}
\]

[Think of \(|x|\) as the instant of time at which \(x\) exists and \(x \mapsto x^+\) as saying how an element evolves from one instant to the next.]

- Morphisms \(f : (X, (\_)^+, [\_]) \to (Y, (\_)^+, [\_])\) are functions \(f \in \text{Set}(X, Y)\) satisfying for all \(x \in X\)

\[
(f x)^+ = f(x^+) \quad \text{(18)}
\]

\[
|f x| = |x| \quad \text{(19)}
\]
• Composition and identities are as in the category Set.

Show that Set has a terminal object and binary products.

7. Show that the category PreOrd of pre-ordered sets and monotone functions is a cartesian closed category.