Randomised Algorithms

Lecture 9-10: Randomised Approximation Algorithms

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Outline

Approximation Ratio for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost (value) $\mathbf{E}[C]$ of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{\mathbf{E}[C]}{C^*}, \frac{C^*}{\mathbf{E}[C]}\right) \leq \rho(n).$$

not covered here...

Randomised Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

MAX-3-CNF Satisfiability

Assume that no literal (including its negation) appears more than once in the same clause.

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

Relaxation of the satisfiability problem. Want to compute how "close" the formula to being satisfiable is.

Example:

$$(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_3} \vee \overline{x_5}) \wedge (x_2 \vee \overline{x_4} \vee x_5) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$$

$x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$ and $x_5 = 1$ satisfies 3 (out of 4 clauses)

Idea: What about assigning each variable uniformly and independently at random?

Analysis

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

Proof:

- For every clause $i = 1, 2, \ldots, m$, define a random variable:

$$Y_i = \mathbf{1}\{\text{clause } i \text{ is satisfied}\}$$

- Since each literal (including its negation) appears at most once in clause i,

$$\mathbf{P}[\,\text{clause } i \text{ is not satisfied}\,] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$$\Rightarrow \quad \mathbf{P}[\,\text{clause } i \text{ is satisfied}\,] = 1 - \frac{1}{8} = \frac{7}{8}$$

$$\Rightarrow \quad \mathbf{E}[\,Y_i\,] = \mathbf{P}[\,Y_i = 1\,] \cdot 1 = \frac{7}{8}.$$

- Let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,

$$\mathbf{E}[\,Y\,] = \mathbf{E}\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} \mathbf{E}[\,Y_i\,] = \sum_{i=1}^{m} \frac{7}{8} = \frac{7}{8} \cdot m. \qquad \square$$

Linearity of Expectations

maximum number of satisfiable clauses is m

Interesting Implications

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

Corollary

For any instance of MAX-3-CNF, there exists an assigment which satisfies at least $\frac{7}{8}$ of all clauses.

There is $\omega \in \Omega$ such that $Y(\omega) \geq \mathbf{E}[\,Y\,]$

Probabilistic Method: powerful tool to show existence of a non-obvious property.

Corollary

Any instance of MAX-3-CNF with at most 7 clauses is satisfiable.

Follows from the previous Corollary.

Expected Approximation Ratio

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

One could prove that the probability to satisfy $(7/8) \cdot m$ clauses is at least $1/(8m)$

$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$

```
GREEDY-3-CNF(φ, n, m)
for j = 1, 2, ..., n
    Compute E[ Y | x_1 = v_1 ..., x_{j-1} = v_{j-1}, x_j = 1 ]
    Compute E[ Y | x_1 = v_1, ..., x_{j-1} = v_{j-1}, x_j = 0 ]
    Let x_j = v_j so that the conditional expectation is maximized
return the assignment v_1, v_2, ..., v_n
```

Analysis of GREEDY-3-CNF(ϕ, n, m)

This algorithm is deterministic.

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Proof:

- **Step 1:** polynomial-time algorithm
  - In iteration $j = 1, 2, \ldots, n$, $Y = Y(\phi)$ averages over 2^{n-j+1} assignments
  - A smarter way is to use linearity of (conditional) expectations:

$$\mathbf{E}\left[\, Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \,\right] = \sum_{i=1}^{m} \mathbf{E}\left[\, Y_i \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = 1 \,\right]$$

computable in $O(1)$

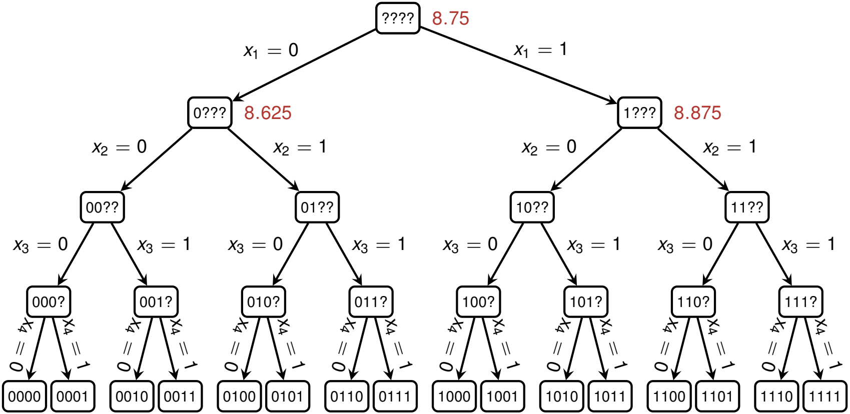
- **Step 2:** satisfies at least $7/8 \cdot m$ clauses
  - Due to the greedy choice in each iteration $j = 1, 2, \ldots, n$,

$$\begin{aligned}
\mathbf{E}\left[\, Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1}, x_j = v_j \,\right] &\geq \mathbf{E}\left[\, Y \mid x_1 = v_1, \ldots, x_{j-1} = v_{j-1} \,\right] \\
&\geq \mathbf{E}\left[\, Y \mid x_1 = v_1, \ldots, x_{j-2} = v_{j-2} \,\right] \\
&\vdots \\
&\geq \mathbf{E}\left[\, Y \,\right] = \frac{7}{8} \cdot m.
\end{aligned}$$

□

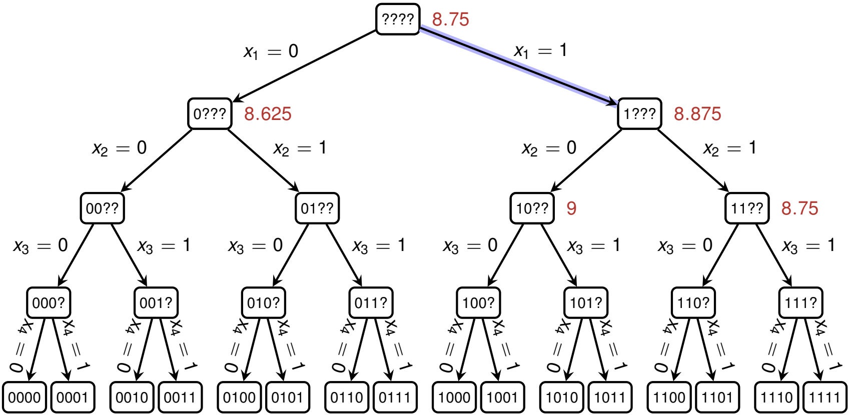
Run of GREEDY-3-CNF(φ, n, m)

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge$$
$$(\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



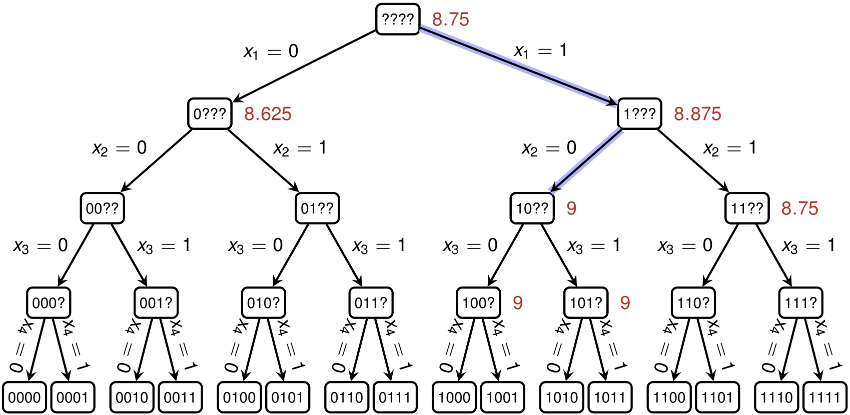
Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



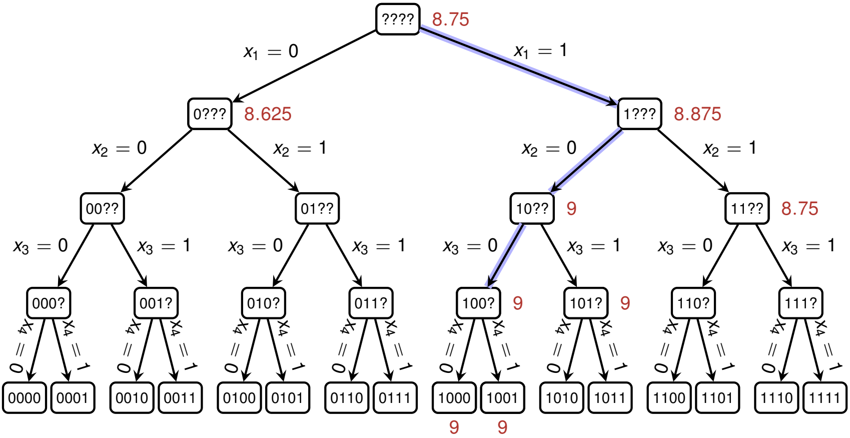
Run of GREEDY-3-CNF(φ, n, m)

$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge 1 \wedge (x_3) \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee \overline{x_4})$$



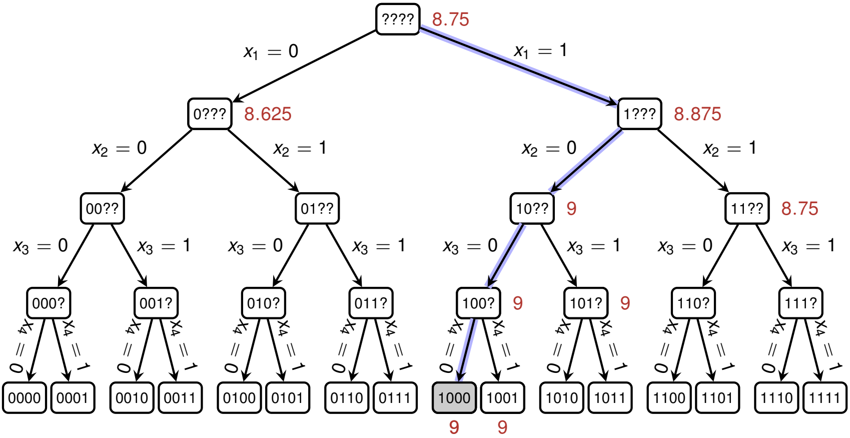
Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



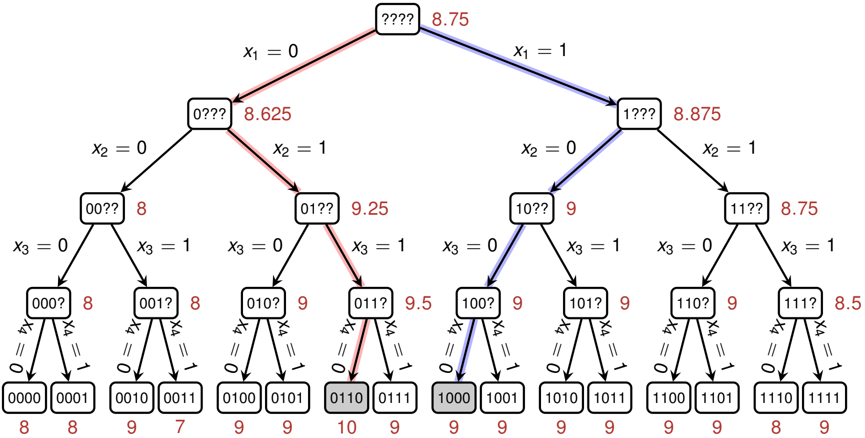
Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor \overline{x_4}) \land (x_1 \lor x_2 \lor \overline{x_4}) \land (\overline{x_1} \lor \overline{x_3} \lor x_4) \land (x_1 \lor x_2 \lor \overline{x_4}) \land$
$(\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \land (\overline{x_1} \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor x_3) \land (x_1 \lor x_3 \lor x_4) \land (x_2 \lor \overline{x_3} \lor \overline{x_4})$



Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

Theorem (Hastad'97)

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.

Essentially there is nothing smarter than just guessing!

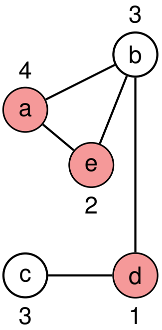
Outline

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.

This is (still) an NP-hard problem.



Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

A Greedy Approach working for Unweighted Vertex Cover

APPROX-VERTEX-COVER(G)

```
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ∪ {u, v}
    remove from E' every edge incident on either u or v
return C
```

This algorithm is a 2-approximation for **unweighted graphs**!

A Greedy Approach working for Unweighted Vertex Cover

```
APPROX-VERTEX-COVER(G)
C = ∅
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return C
```
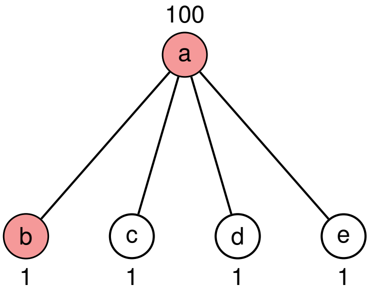
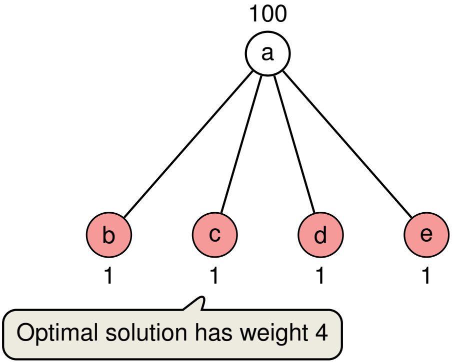



Computed solution has weight 101

A Greedy Approach working for Unweighted Vertex Cover

APPROX-VERTEX-COVER(G)

```
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ∪ {u, v}
    remove from E' every edge incident on either u or v
return C
```



Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

0-1 Integer Program

$$\begin{aligned}\text{minimize} \quad & \sum_{v\in V} w(v)x(v) \\ \text{subject to} \quad & x(u)+x(v) \geq 1 && \text{for each } (u,v)\in E \\ & x(v) \in \{0,1\} && \text{for each } v \in V\end{aligned}$$

optimum is a lower bound on the optimal weight of a minimum weight-cover.

Linear Program

$$\begin{aligned}\text{minimize} \quad & \sum_{v\in V} w(v)x(v) \\ \text{subject to} \quad & x(u)+x(v) \geq 1 && \text{for each } (u,v)\in E \\ & x(v) \in [0,1] && \text{for each } v \in V\end{aligned}$$

Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

```
C = ∅
compute x̄, an optimal solution to the linear program
for each v ∈ V
    if x̄(v) ≥ 1/2
        C = C ∪ {v}
return C
```

Theorem 35.7

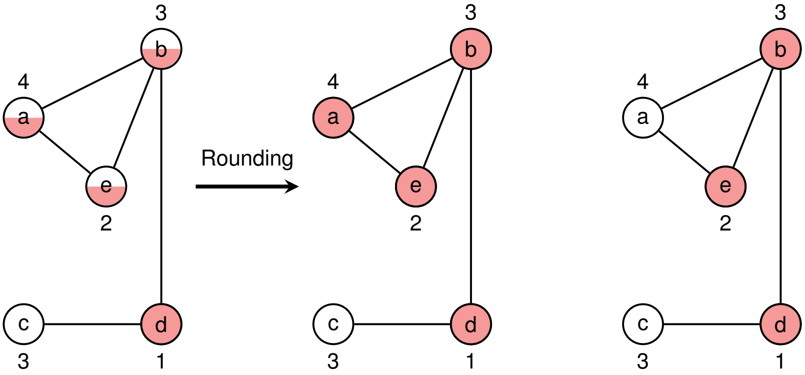
APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time

Example of Approx-Min-Weight-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}, \overline{x}(d) = 1, \overline{x}(c) = 0$

$x(a) = x(b) = x(e) = 1, x(d) = 1, x(c) = 0$



fractional solution of LP with weight $= 5.5$

rounded solution of LP with weight $= 10$

optimal solution with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

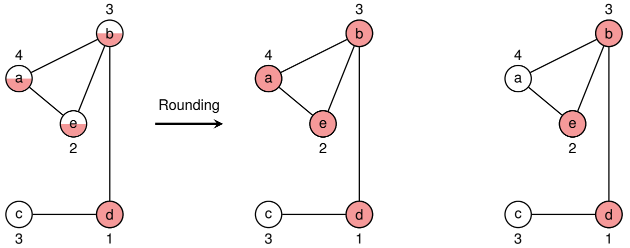
- Let C^* be an optimal solution to the minimum-weight vertex cover problem
- Let z^* be the value of an optimal solution to the linear program, so

$$z^* \leq w(C^*)$$

- **Step 1:** The computed set C covers all vertices:
  - Consider any edge $(u, v) \in E$ which imposes the constraint $x(u) + x(v) \geq 1$

  $\Rightarrow$ at least one of $\overline{x}(u)$ and $\overline{x}(v)$ is at least $1/2$ $\Rightarrow$ C covers edge (u, v)
- **Step 2:** The computed set C satisfies $w(C) \leq 2z^*$:

$$w(C^*) \geq z^* = \sum_{v \in V} w(v)\overline{x}(v) \geq \sum_{v \in V:\ \overline{x}(v) \geq 1/2} w(v) \cdot \frac{1}{2} = \frac{1}{2} w(C). \quad \square$$



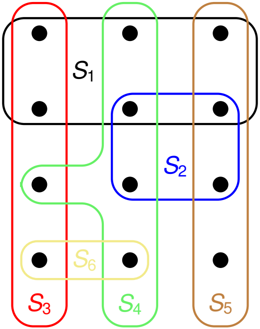
Outline

The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$ s.t. $X = \bigcup_{S \in \mathcal{C}} S.$

Sum over the costs of all sets in $\mathcal{C}$



	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2

Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

Setting up an Integer Program

Exercise: Try to formulate the integer program and linear program of the weighted SET-COVER problem (solution on next slide!)

Setting up an Integer Program

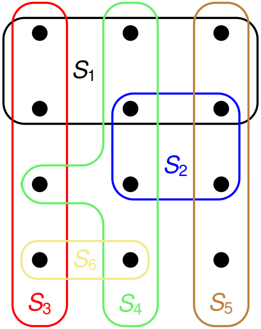
0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\
\text{subject to} & \displaystyle\sum_{S \in \mathcal{F}\colon x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\
 & y(S) & \in & \{0, 1\} & \text{for each } S \in \mathcal{F}
\end{array}
$$

Linear Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\
\text{subject to} & \displaystyle\sum_{S \in \mathcal{F}\colon x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\
 & y(S) & \in & [0, 1] & \text{for each } S \in \mathcal{F}
\end{array}
$$

Back to the Example



	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2
$\overline{y}(.)$:	$1/2$	$1/2$	$1/2$	$1/2$	1	$1/2$

Cost equals 8.5

The strategy employed for Vertex-Cover would take all 6 sets!

Even worse: If all $\overline{y}$'s were below $1/2$, we would not even return a valid cover!

Randomised Rounding

	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2
$\overline{y}(.)$:	$1/2$	$1/2$	$1/2$	$1/2$	1	$1/2$

Idea: Interpret the $\overline{y}$-values as probabilities for picking the respective set.

Randomised Rounding

- Let $\mathcal{C} \subseteq \mathcal{F}$ be a random set with each set S being included independently with probability $\overline{y}(S)$.
- More precisely, if $\overline{y}$ denotes the optimal solution of the LP, then we compute an integral solution y by:

$$y(S) = \begin{cases} 1 & \text{with probability } \overline{y}(S) \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for all } S \in \mathcal{F}.$$

- Therefore, $\mathbf{E}\left[\, y(S) \,\right] = \overline{y}(S)$.

Randomised Rounding

	S_1	S_2	S_3	S_4	S_5	S_6
c :	2	3	3	5	1	2
$\overline{y}(.)$:	1/2	1/2	1/2	1/2	1	1/2

Idea: Interpret the $\overline{y}$-values as probabilities for picking the respective set.

Lemma

- The expected cost satisfies

$$\mathbf{E}\left[\, c(\mathcal{C}) \,\right] = \sum_{S \in \mathcal{F}} c(S) \cdot \overline{y}(S)$$

- The probability that an element $x \in X$ is covered satisfies

$$\mathbf{P}\left[x \in \bigcup_{S \in \mathcal{C}} S \right] \geq 1 - \frac{1}{e}.$$

Proof of Lemma

Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $\overline{y}(S)$.

- The expected cost satisfies $\mathbf{E}[\, c(\mathcal{C})\,] = \sum_{S \in \mathcal{F}} c(S) \cdot \overline{y}(S)$.
- The probability that x is covered satisfies $\mathbf{P}[\, x \in \cup_{S \in \mathcal{C}} S\,] \geq 1 - \frac{1}{e}$.

Proof:

- **Step 1**: The expected cost of the random set $\mathcal{C}$

$$\mathbf{E}[\, c(\mathcal{C})\,] = \mathbf{E}\left[\sum_{S \in \mathcal{C}} c(S)\right] = \mathbf{E}\left[\sum_{S \in \mathcal{F}} \mathbf{1}_{S \in \mathcal{C}} \cdot c(S)\right]$$

$$= \sum_{S \in \mathcal{F}} \mathbf{P}[\, S \in \mathcal{C}\,] \cdot c(S) = \sum_{S \in \mathcal{F}} \overline{y}(S) \cdot c(S).$$

- **Step 2**: The probability for an element to be (not) covered

$$\mathbf{P}[\, x \notin \cup_{S \in \mathcal{C}} S\,] = \prod_{S \in \mathcal{F}\colon x \in S} \mathbf{P}[\, S \notin \mathcal{C}\,] = \prod_{S \in \mathcal{F}\colon x \in S} (1 - \overline{y}(S))$$

$$\leq \prod_{S \in \mathcal{F}\colon x \in S} e^{-\overline{y}(S)}$$

$$= e^{-\sum_{S \in \mathcal{F}\colon x \in S} \overline{y}(S)} \leq e^{-1} \quad \square$$

$1 + x \leq e^x$ for any $x \in \mathbb{R}$

$\overline{y}$ solves the LP!

The Final Step

Lemma

Let $\mathcal{C} \subseteq \mathcal{F}$ be a random subset with each set S being included independently with probability $y(S)$.

- The expected cost satisfies $\mathbf{E}[\, c(\mathcal{C}) \,] = \sum_{S \in \mathcal{F}} c(S) \cdot y(S)$.
- The probability that x is covered satisfies $\mathbf{P}[\, x \in \cup_{S \in \mathcal{C}} S \,] \geq 1 - \frac{1}{e}$.

Problem: Need to make sure that every element is covered!

Idea: Amplify this probability by taking the union of $\Omega(\log n)$ random sets $\mathcal{C}$.

```
WEIGHTED SET COVER-LP(X, F, c)
compute ȳ, an optimal solution to the linear program
C = ∅
repeat 2 ln n times
    for each S ∈ F
        let C = C ∪ {S} with probability ȳ(S)
return C
```

clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
- The expected approximation ratio is $2\ln(n)$.

Proof:

- **Step 1**: The probability that $\mathcal{C}$ is a cover
  - By previous Lemma, an element $x \in X$ is covered in one of the $2\ln n$ iterations with probability at least $1 - \frac{1}{e}$, so that

$$\mathbf{P}\left[x \notin \cup_{S\in\mathcal{C}} S\right] \leq \left(\frac{1}{e}\right)^{2\ln n} = \frac{1}{n^2}.$$

  - This implies for the event that all elements are covered:

$$\mathbf{P}\left[X = \cup_{S\in\mathcal{C}} S\right] = 1 - \mathbf{P}\left[\bigcup_{x\in X}\{x \notin \cup_{S\in\mathcal{C}} S\}\right]$$

$\mathbf{P}[A \cup B] \leq \mathbf{P}[A] + \mathbf{P}[B]$

$$\geq 1 - \sum_{x\in X} \mathbf{P}\left[x \notin \cup_{S\in\mathcal{C}} S\right] \geq 1 - n \cdot \frac{1}{n^2} = 1 - \frac{1}{n}.$$

- **Step 2**: The expected approximation ratio
  - By previous lemma, the expected cost of one iteration is $\sum_{S\in\mathcal{F}} c(S) \cdot \overline{y}(S)$.
  - Linearity $\Rightarrow$ $\mathbf{E}\left[c(\mathcal{C})\right] \leq 2\ln(n) \cdot \sum_{S\in\mathcal{F}} c(S) \cdot \overline{y}(S) \leq 2\ln(n) \cdot c(\mathcal{C}^*)$ $\square$

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
- The expected approximation ratio is $2\ln(n)$.

By Markov's inequality, $\mathbf{P}\left[\, c(\mathcal{C}) \leq 4\ln(n) \cdot c(\mathcal{C}^*) \,\right] \geq 1/2$.

Hence with probability at least $1 - \frac{1}{n} - \frac{1}{2} > \frac{1}{3}$, solution is within a factor of $4\ln(n)$ of the optimum.

probability could be further increased by repeating

Typical Approach for Designing Approximation Algorithms based on LPs

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

MAX-CNF

Recall:

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

MAX-CNF Satisfiability (MAX-SAT)

- Given: CNF formula, e.g.: $(x_1 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee x_4 \vee \overline{x_5}) \wedge \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

Why study this generalised problem?

- Allowing arbitrary clause lengths makes the problem more interesting (we will see that simply guessing is not the best!)
- a nice concluding example where we can practice previously learned approaches

Approach 1: Guessing the Assignment

Assign each variable true or false uniformly and independently at random.

Recall: This was the successful approach to solve MAX-3-CNF!

Analysis

For any clause i which has length ℓ,

$$\mathbf{P}[\text{clause } i \text{ is satisfied}] = 1 - 2^{-\ell} := \alpha_\ell.$$

In particular, the guessing algorithm is a randomised 2-approximation.

Proof:

- First statement as in the proof of Theorem 35.6. For clause i not to be satisfied, all ℓ occurring variables must be set to a specific value.
- As before, let $Y := \sum_{i=1}^{m} Y_i$ be the number of satisfied clauses. Then,

$$\mathbf{E}[Y] = \mathbf{E}\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} \mathbf{E}[Y_i] \geq \sum_{i=1}^{m} \frac{1}{2} = \frac{1}{2} \cdot m. \qquad \square$$

Approach 2: Guessing with a "Hunch" (Randomised Rounding)

First solve a linear program and use fractional values for a **biased** coin flip.

The same as randomised rounding!

0-1 Integer Program

$$\begin{array}{lrcll} \text{maximize} & \displaystyle\sum_{i=1}^{m} z_i & & & \\ \text{subject to} & \displaystyle\sum_{j \in C_i^+} y_j + \sum_{j \in C_i^-} (1 - y_j) & \geq & z_i & \text{for each } i = 1, 2, \ldots, m \\ & z_i & \in & \{0, 1\} & \text{for each } i = 1, 2, \ldots, m \\ & y_j & \in & \{0, 1\} & \text{for each } j = 1, 2, \ldots, n \end{array}$$

These auxiliary variables are used to reflect whether a clause is satisfied or not

C_i^+ is the index set of the un-negated variables of clause i.

- In the corresponding LP each $\in \{0, 1\}$ is replaced by $\in [0, 1]$
- Let $(\overline{y}, \overline{z})$ be the optimal solution of the LP
- Obtain an integer solution y through randomised rounding of $\overline{y}$

Analysis of Randomised Rounding

Lemma

For any clause i of length ℓ,

$$\mathbf{P}\left[\text{clause } i \text{ is satisfied}\right] \geq \left(1 - \left(1 - \frac{1}{\ell}\right)^{\ell}\right) \cdot \overline{z}_i.$$

Proof of Lemma (1/2):

- Assume w.l.o.g. all literals in clause i appear non-negated (otherwise replace every occurrence of x_j by $\overline{x_j}$ in the whole formula)
- Further, by relabelling assume $C_i = (x_1 \vee \cdots \vee x_\ell)$

$$\Rightarrow \quad \mathbf{P}\left[\text{clause } i \text{ is satisfied}\right] = 1 - \prod_{j=1}^{\ell} \mathbf{P}\left[\ y_j \text{ is false }\ \right] = 1 - \prod_{j=1}^{\ell}(1 - \overline{y}_j)$$

$$\geq 1 - \left(\frac{\sum_{j=1}^{\ell}(1 - \overline{y}_j)}{\ell}\right)^{\ell}$$

$$= 1 - \left(1 - \frac{\sum_{j=1}^{\ell} \overline{y}_j}{\ell}\right)^{\ell} \geq 1 - \left(1 - \frac{\overline{z}_i}{\ell}\right)^{\ell}.$$

Arithmetic vs. geometric mean:

$$\frac{a_1 + \ldots + a_k}{k} \geq \sqrt[k]{a_1 \times \ldots \times a_k}.$$

Analysis of Randomised Rounding

Lemma

For any clause i of length ℓ,

$$\mathbf{P}\left[\,\text{clause } i \text{ is satisfied}\,\right] \geq \left(1 - \left(1 - \frac{1}{\ell}\right)^{\ell}\right) \cdot \overline{z}_i.$$

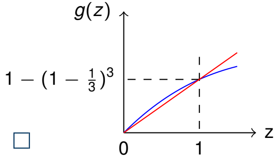
Proof of Lemma (2/2):

- So far we have shown:

$$\mathbf{P}\left[\,\text{clause } i \text{ is satisfied}\,\right] \geq 1 - \left(1 - \frac{\overline{z}_i}{\ell}\right)^{\ell}$$

- For any $\ell \geq 1$, define $g(z) := 1 - \left(1 - \frac{z}{\ell}\right)^{\ell}$. This is a concave function with $g(0) = 0$ and $g(1) = 1 - \left(1 - \frac{1}{\ell}\right)^{\ell} =: \beta_\ell$.

$$\Rightarrow \quad g(z) \geq \beta_\ell \cdot z \qquad \text{for any } z \in [0, 1]$$



- Therefore, $\mathbf{P}\left[\,\text{clause } i \text{ is satisfied}\,\right] \geq \beta_\ell \cdot \overline{z}_i$. □

Analysis of Randomised Rounding

Lemma

For any clause i of length ℓ,

$$\mathbf{P}\left[\,\text{clause } i \text{ is satisfied}\,\right] \geq \left(1 - \left(1 - \frac{1}{\ell}\right)^{\ell}\right) \cdot \overline{z}_i.$$

Theorem

Randomised Rounding yields a $1/(1 - 1/e) \approx 1.5820$ randomised approximation algorithm for MAX-CNF.

Proof of Theorem:

- For any clause $i = 1, 2, \ldots, m$, let ℓ_i be the corresponding length.
- Then the expected number of satisfied clauses is:

$$\mathbf{E}\left[\,Y\,\right] = \sum_{i=1}^{m} \mathbf{E}\left[\,Y_i\,\right] \geq \sum_{i=1}^{m} \left(1 - \left(1 - \frac{1}{\ell_i}\right)^{\ell_i}\right) \cdot \overline{z}_i \geq \sum_{i=1}^{m} \left(1 - \frac{1}{e}\right) \cdot \overline{z}_i \geq \left(1 - \frac{1}{e}\right) \cdot \text{OPT}$$

By Lemma

Since $(1 - 1/x)^x \leq 1/e$

LP solution at least as good as optimum

Approach 3: Hybrid Algorithm

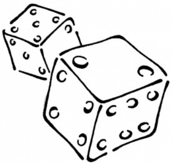
Summary

- Approach 1 (Guessing) achieves better guarantee on longer clauses
- Approach 2 (Rounding) achieves better guarantee on shorter clauses

Idea: Consider a hybrid algorithm which interpolates between the two approaches

HYBRID-MAX-CNF(φ, n, m)

Let $b \in \{0, 1\}$ be the flip of a fair coin
If $b = 0$ **then** perform random guessing
If $b = 1$ **then** perform randomised rounding
return the computed solution



Algorithm sets each variable x_i to TRUE with prob. $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \overline{y}_i$.
Note, however, that variables are **not** independently assigned!

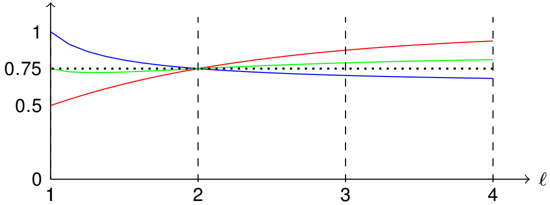
Analysis of Hybrid Algorithm

Theorem

HYBRID-MAX-CNF(φ, n, m) is a randomised 4/3-approx. algorithm.

Proof:

- It suffices to prove that clause i is satisfied with probability at least $3/4 \cdot \overline{z}_i$
- For any clause i of length ℓ:
  - Algorithm 1 satisfies it with probability $1 - 2^{-\ell} = \alpha_\ell \geq \alpha_\ell \cdot \overline{z}_i$.
  - Algorithm 2 satisfies it with probability $\beta_\ell \cdot \overline{z}_i$.
  - HYBRID-MAX-CNF(φ, n, m) satisfies it with probability $\frac{1}{2} \cdot \alpha_\ell \cdot \overline{z}_i + \frac{1}{2} \cdot \beta_\ell \cdot \overline{z}_i$.
- Note $\frac{\alpha_\ell + \beta_\ell}{2} = 3/4$ for $\ell \in \{1, 2\}$, and for $\ell \geq 3$, $\frac{\alpha_\ell + \beta_\ell}{2} \geq 3/4$ (see figure)
- $\Rightarrow$ HYBRID-MAX-CNF(φ, n, m) satisfies it with prob. at least $3/4 \cdot \overline{z}_i$ □



MAX-CNF Conclusion

Summary

- Since $\alpha_2 = \beta_2 = 3/4$, we cannot achieve a better approximation ratio than $4/3$ by combining Algorithm 1 & 2 in a different way
- The $4/3$-approximation algorithm can be easily derandomised
  - Idea: use the conditional expectation trick for both Algorithm 1 & 2 and output the better solution
- The $4/3$-approximation algorithm applies unchanged to a weighted version of MAX-CNF, where each clause has a non-negative weight
- Even MAX-2-CNF (every clause has length 2) is NP-hard!