Randomised Algorithms

Lecture 9-10: Randomised Approximation Algorithms

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Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

Approximation Ratio for Randomised Approximation Algorithms

Approximation Ratio

A randomised algorithm for a problem has approximation ratio $\rho(n)$, if for any input of size n, the expected cost (value) $\mathbf{E}[C]$ of the returned solution and optimal cost C^* satisfy:

$$\max\left(\frac{\mathbf{E}[C]}{C^*}, \frac{C^*}{\mathbf{E}[C]}\right) \leq \rho(n).$$

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- Maximization problem: $\frac{C^*}{\mathbf{E}[C]} \ge 1$
- Minimization problem: $\frac{\mathbf{E}[C]}{C^*} \ge 1$

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Randomised Approximation Schemes

An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

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An approximation scheme is an approximation algorithm, which given any input and $\epsilon > 0$, is a $(1+\epsilon)$-approximation algorithm.

- It is a polynomial-time approximation scheme (PTAS) if for any fixed $\epsilon > 0$, the runtime is polynomial in n. For example, $O(n^{2/\epsilon})$.
- It is a fully polynomial-time approximation scheme (FPTAS) if the runtime is polynomial in both $1/\epsilon$ and n. For example, $O((1/\epsilon)^2 \cdot n^3)$.

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Example:

$$(x_1 \lor x_3 \lor \overline{x_4}) \land (x_1 \lor \overline{x_3} \lor \overline{x_5}) \land (x_2 \lor \overline{x_4} \lor x_5) \land (\overline{x_1} \lor x_2 \lor \overline{x_3})$$

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Idea: What about assigning each variable uniformly and independently at random?

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Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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Follows from the previous Corollary.

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Y is defined as in the previous proof.

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Algorithm: Assign x_1 so that the conditional expectation is maximized and recurse.

Expected Approximation Ratio

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a polynomial-time randomised $8/7$-approximation algorithm.

One could prove that the probability to satisfy $(7/8) \cdot m$ clauses is at least $1/(8m)$

$$\mathbf{E}[\,Y\,] = \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 1\,] + \frac{1}{2} \cdot \mathbf{E}[\,Y \mid x_1 = 0\,].$$

Y is defined as in the previous proof.

One of the two conditional expectations is at least $\mathbf{E}[\,Y\,]$

```
GREEDY-3-CNF(φ, n, m)
for j = 1, 2, ..., n
    Compute E[ Y | x_1 = v_1 ..., x_{j-1} = v_{j-1}, x_j = 1 ]
    Compute E[ Y | x_1 = v_1, ..., x_{j-1} = v_{j-1}, x_j = 0 ]
    Let x_j = v_j so that the conditional expectation is maximized
return the assignment v_1, v_2, ..., v_n
```

Analysis of GREEDY-3-CNF(ϕ, n, m)

Theorem

GREEDY-3-CNF(ϕ, n, m) is a polynomial-time $8/7$-approximation.

This algorithm is deterministic.

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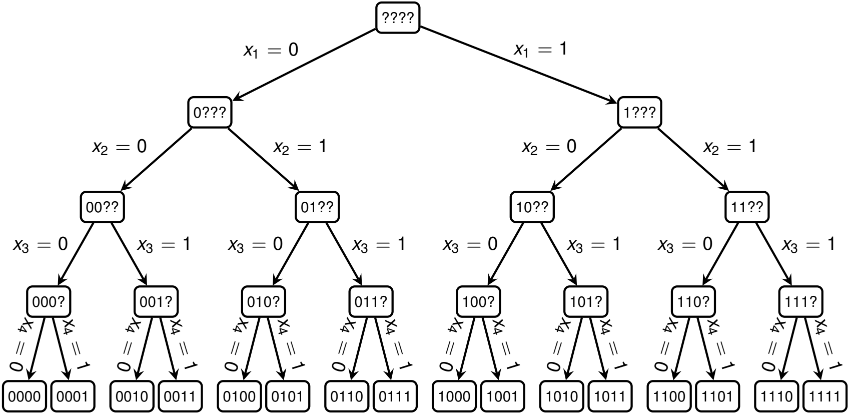
Run of GREEDY-3-CNF(φ, n, m)

$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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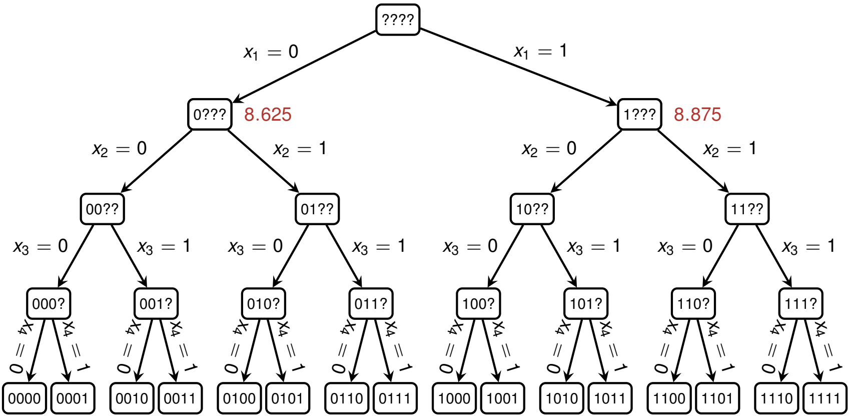
$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$



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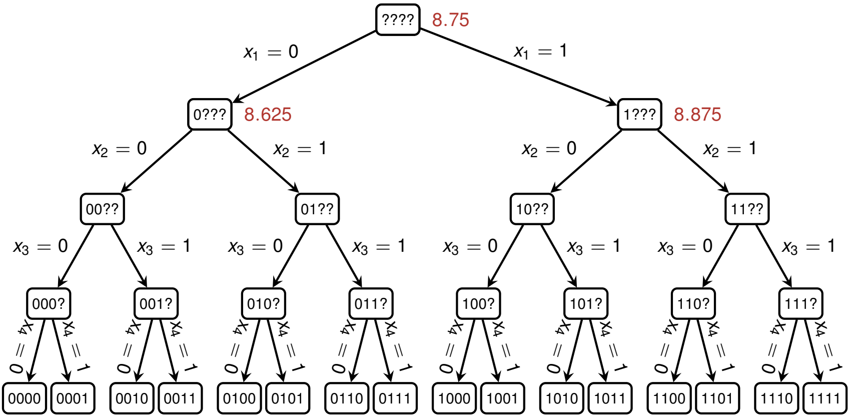

Run of GREEDY-3-CNF(φ, n, m)

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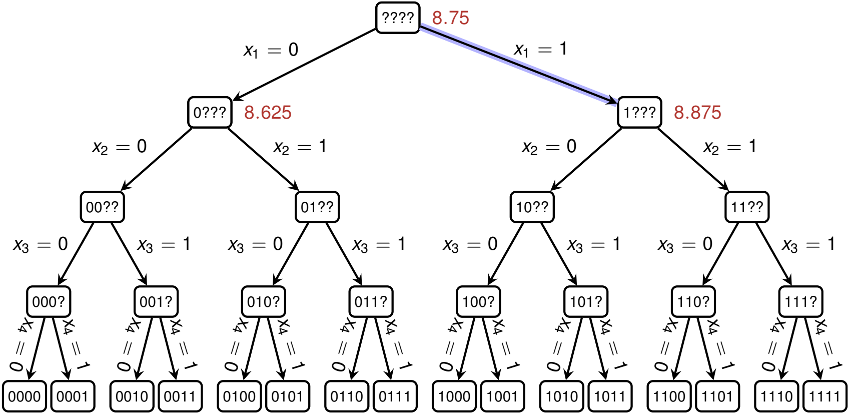

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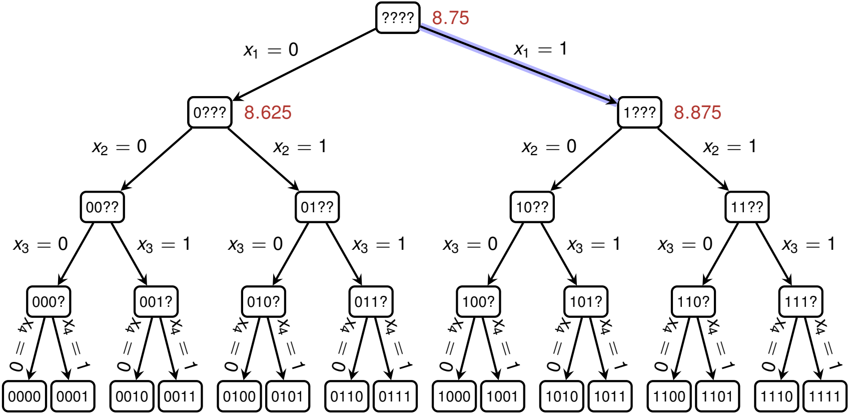
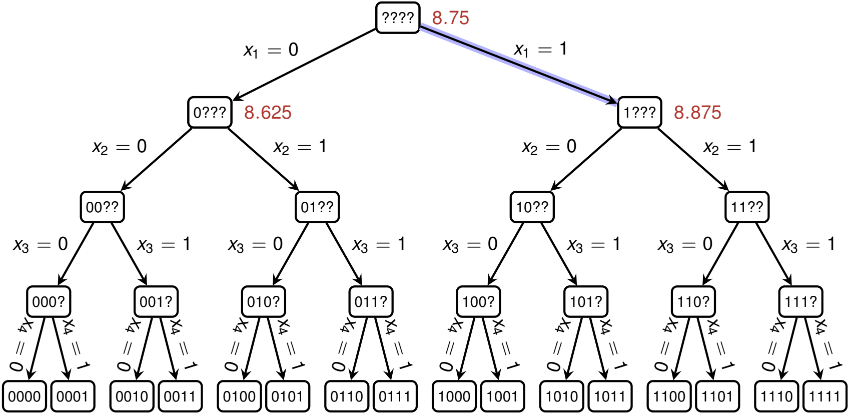


$$1 \wedge 1 \wedge 1 \wedge (\overline{x_3} \vee x_4) \wedge 1 \wedge (\overline{x_2} \vee \overline{x_3}) \wedge (x_2 \vee x_3) \wedge (\overline{x_2} \vee x_3) \wedge 1 \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



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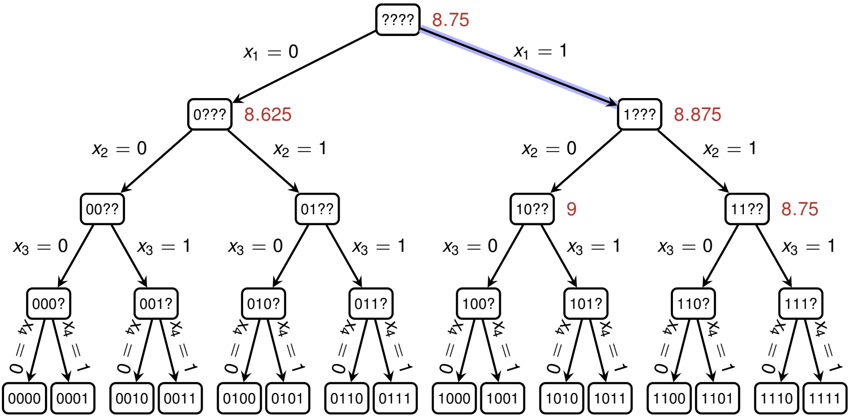

Run of GREEDY-3-CNF(φ, n, m)

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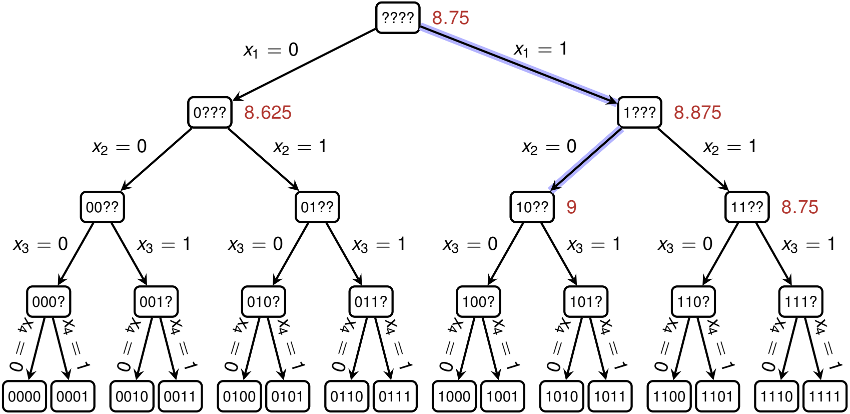

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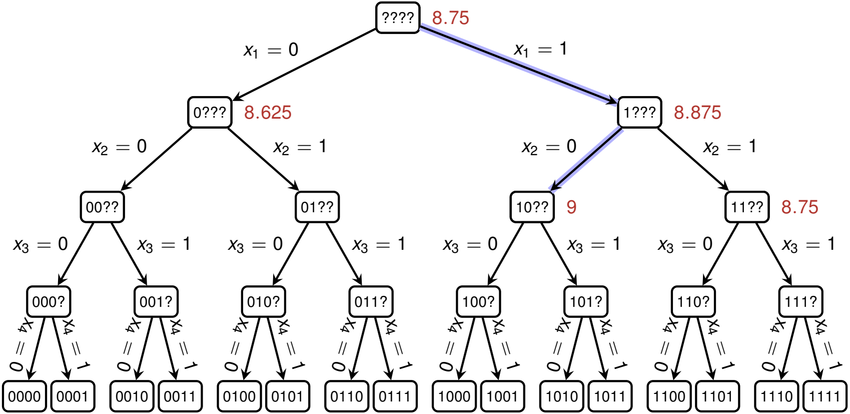

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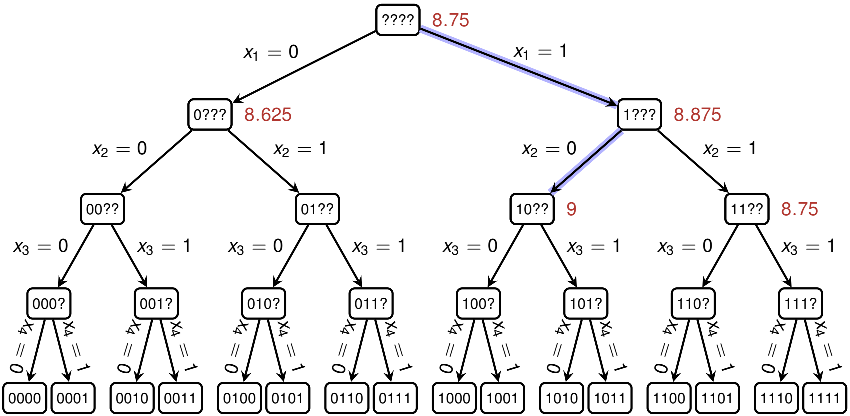

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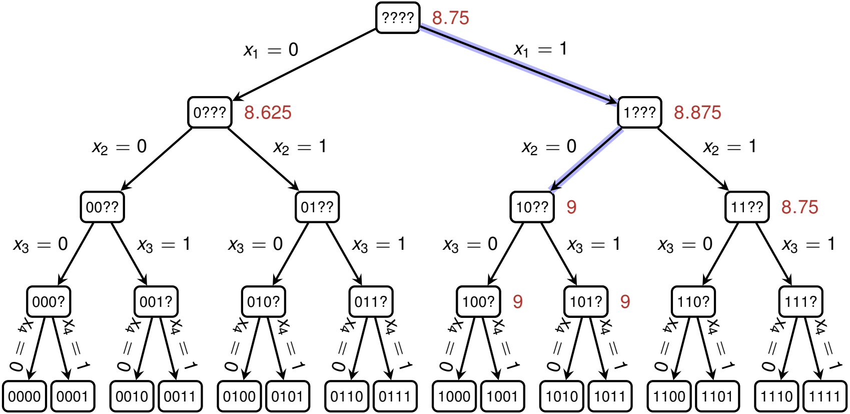

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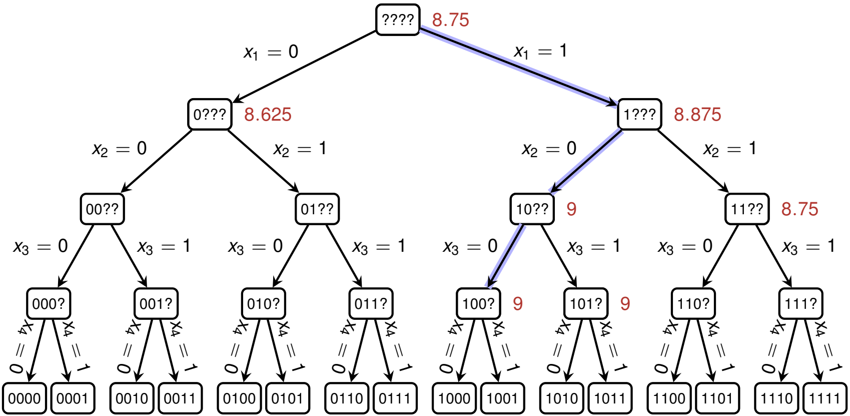

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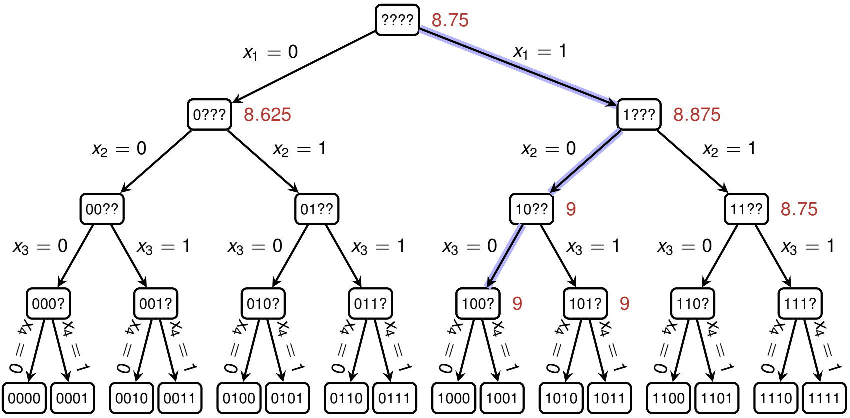
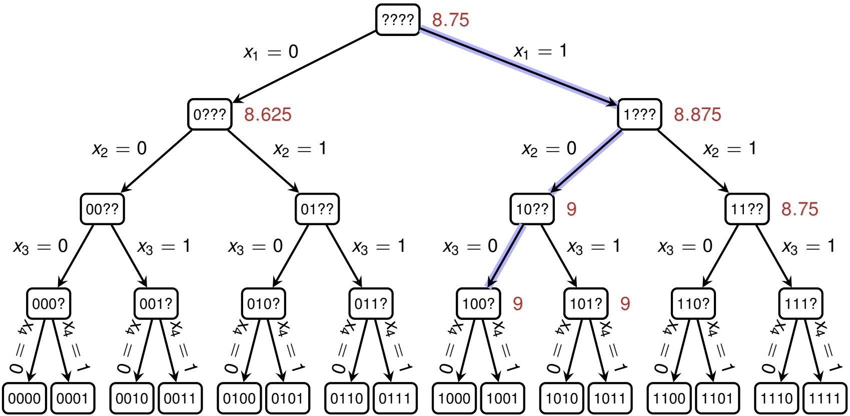


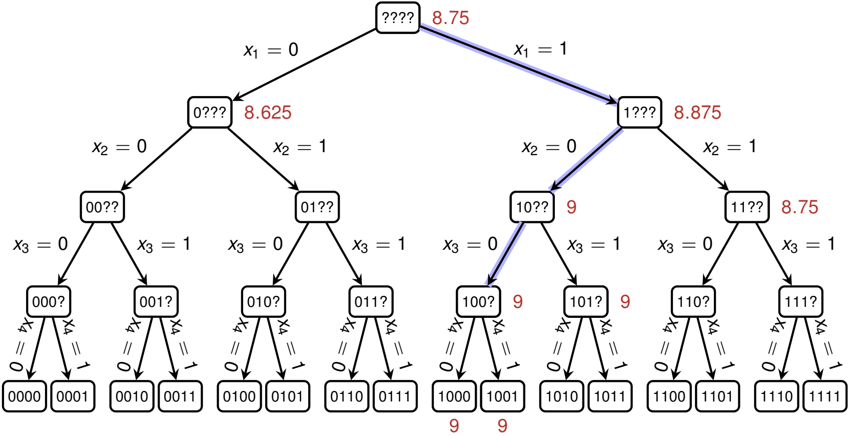
Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



Run of GREEDY-3-CNF(φ, n, m)

$1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 \wedge 0 \wedge 1 \wedge 1 \wedge 1$



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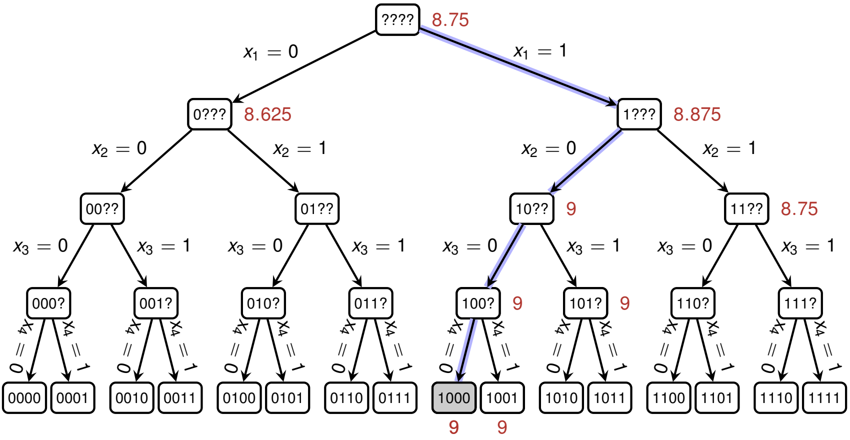

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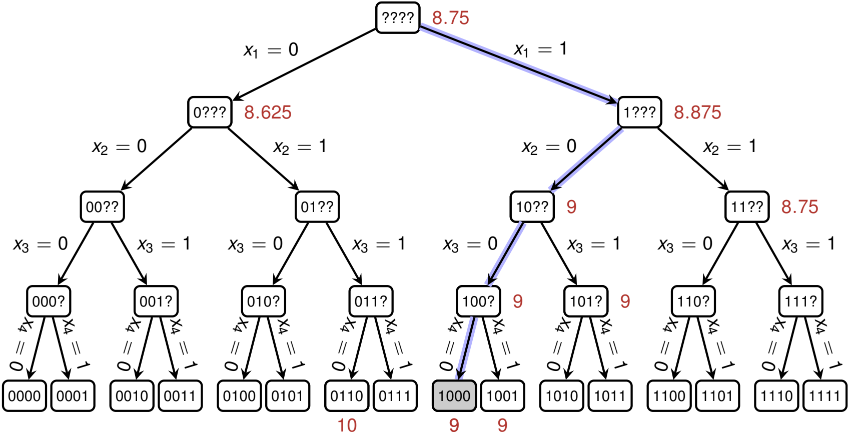

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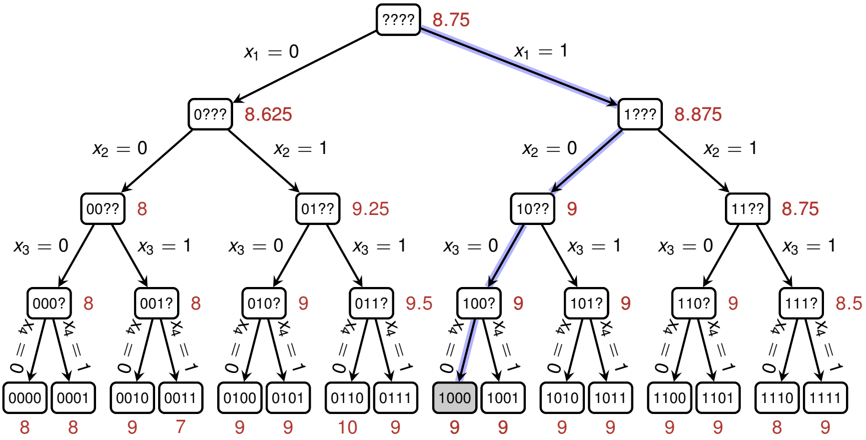

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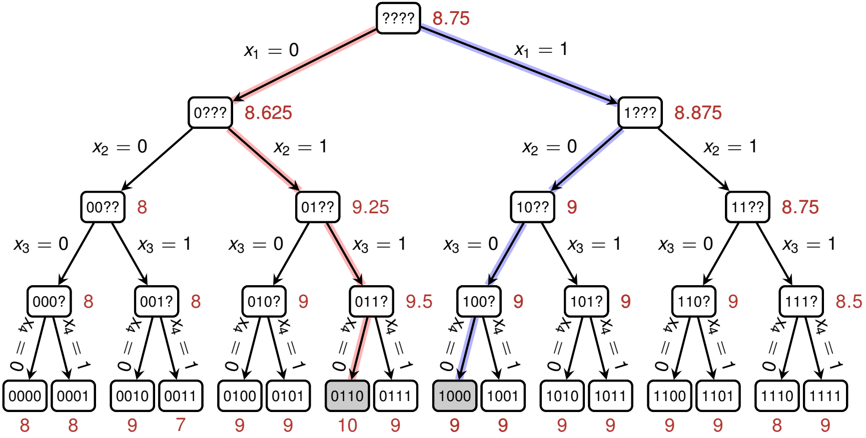

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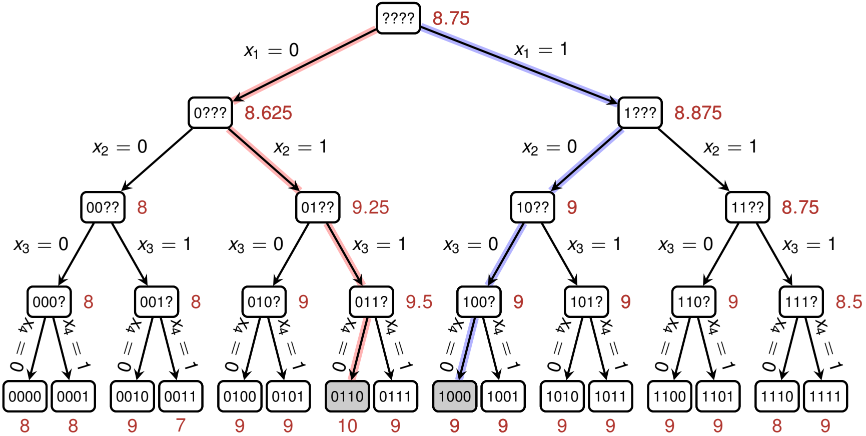

Returned solution satisfies 9 out of 10 clauses, but the formula is satisfiable.

MAX-3-CNF: Concluding Remarks

Theorem 35.6

Given an instance of MAX-3-CNF with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomised algorithm that sets each variable independently at random is a randomised $8/7$-approximation algorithm.

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GREEDY-3-CNF(ϕ, n, m) is a polynomial-time 8/7-approximation.

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Theorem (Hastad'97)

For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.

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For any $\epsilon > 0$, there is no polynomial time $8/7 - \epsilon$ approximation algorithm of MAX3-CNF unless P=NP.

Essentially there is nothing smarter than just guessing!

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

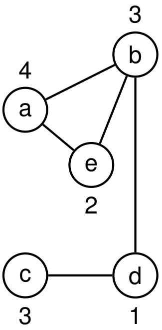
Weighted Set Cover

MAX-CNF

The Weighted Vertex-Cover Problem

Vertex Cover Problem

- Given: Undirected, vertex-weighted graph $G = (V, E)$
- Goal: Find a minimum-weight subset $V' \subseteq V$ such that if $(u, v) \in E(G)$, then $u \in V'$ or $v \in V'$.



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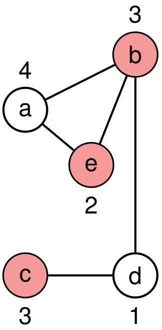

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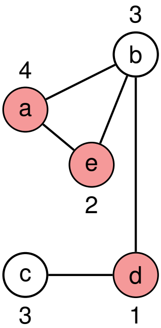

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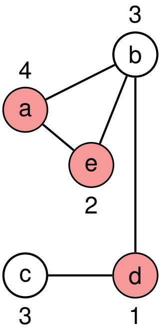

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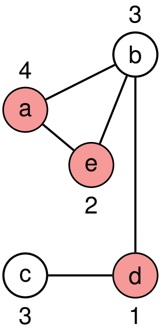

Applications:

The Weighted Vertex-Cover Problem

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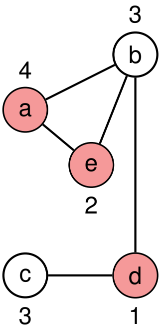

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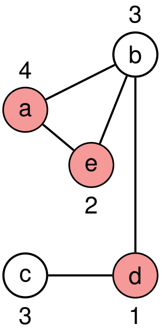

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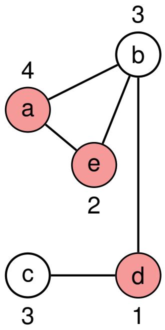

Applications:

- Every edge forms a task, and every vertex represents a person/machine which can execute that task
- Weight of a vertex could be salary of a person
- Perform all tasks with the minimal amount of resources

A Greedy Approach working for Unweighted Vertex Cover

```
APPROX-VERTEX-COVER(G)
C = ∅
E' = G.E
while E' ≠ ∅
    let (u, v) be an arbitrary edge of E'
    C = C ∪ {u, v}
    remove from E' every edge incident on either u or v
return C
```

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This algorithm is a 2-approximation for **unweighted graphs**!

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while E′ ≠ ∅
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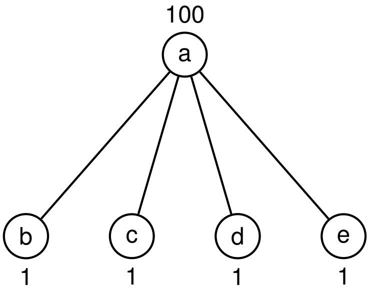

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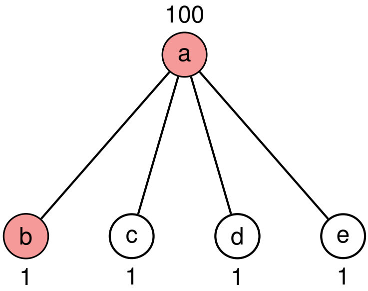
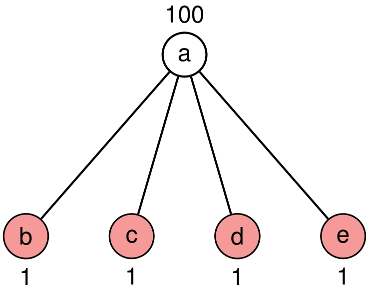


Computed solution has weight 101

A Greedy Approach working for Unweighted Vertex Cover

APPROX-VERTEX-COVER(G)

$C = \emptyset$
$E' = G.E$
while $E' \neq \emptyset$
    let (u, v) be an arbitrary edge of E'
    $C = C \cup \{u, v\}$
    remove from E' every edge incident on either u or v
return C



Optimal solution has weight 4

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

Invoking an (Integer) Linear Program

Idea: Round the solution of an associated linear program.

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{v \in V} w(v)x(v) & & & \\
\text{subject to} & x(u) + x(v) & \geq & 1 & \text{for each } (u, v) \in E \\
 & x(v) & \in & \{0, 1\} & \text{for each } v \in V
\end{array}
$$

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Linear Program

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optimum is a lower bound on the optimal weight of a minimum weight-cover.

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Rounding Rule: if $x(v) \geq 1/2$ then round up, otherwise round down.

```
APPROX-MIN-WEIGHT-VC(G, w)
C = ∅
compute x̄, an optimal solution to the linear program
for each v ∈ V
    if x̄(v) ≥ 1/2
        C = C ∪ {v}
return C
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Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

The Algorithm

APPROX-MIN-WEIGHT-VC(G, w)

$C = \emptyset$
compute $\bar{x}$, an optimal solution to the linear program
for each $v \in V$
    if $\bar{x}(v) \geq 1/2$
        $C = C \cup \{v\}$
return C

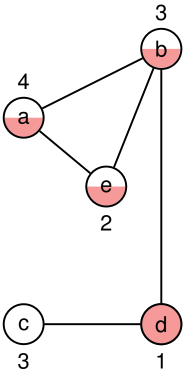
Theorem 35.7

APPROX-MIN-WEIGHT-VC is a polynomial-time 2-approximation algorithm for the minimum-weight vertex-cover problem.

is polynomial-time because we can solve the linear program in polynomial time

Example of APPROX-MIN-WEIGHT-VC

$\overline{x}(a) = \overline{x}(b) = \overline{x}(e) = \frac{1}{2}$, $\overline{x}(d) = 1$, $\overline{x}(c) = 0$



fractional solution of LP
with weight $= 5.5$

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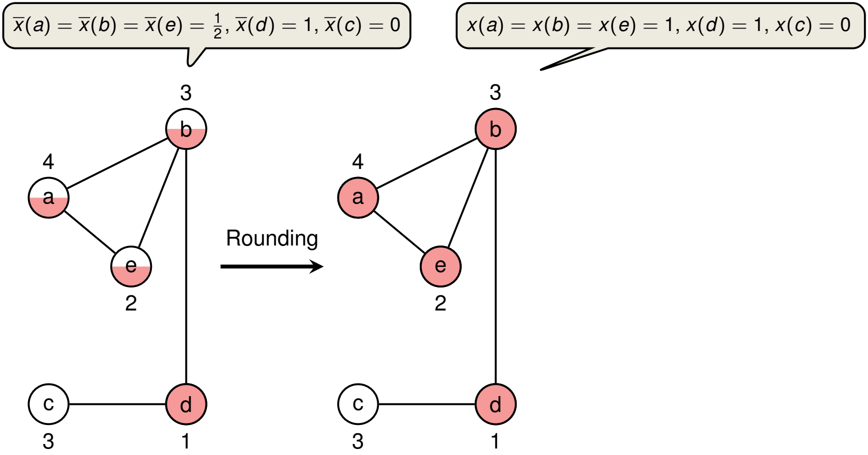

fractional solution of LP
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rounded solution of LP
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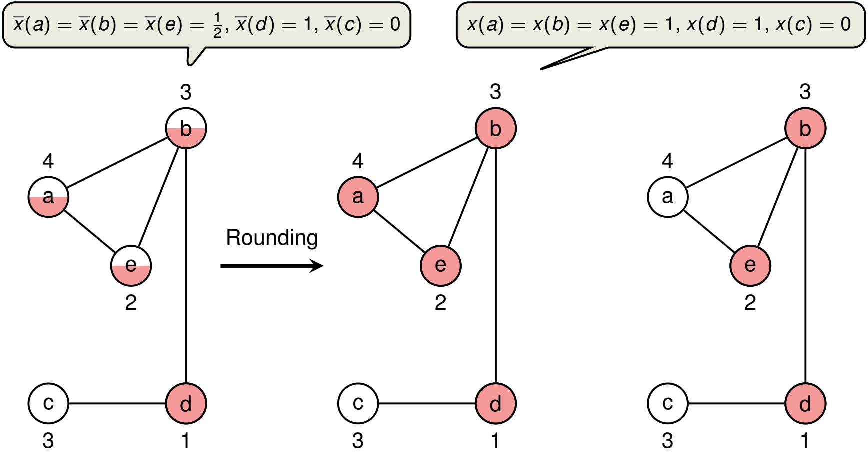

fractional solution of LP with weight $= 5.5$

rounded solution of LP with weight $= 10$

optimal solution with weight $= 6$

Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

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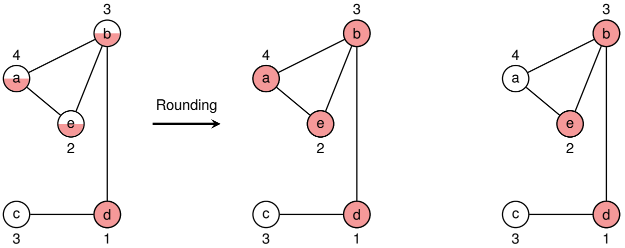

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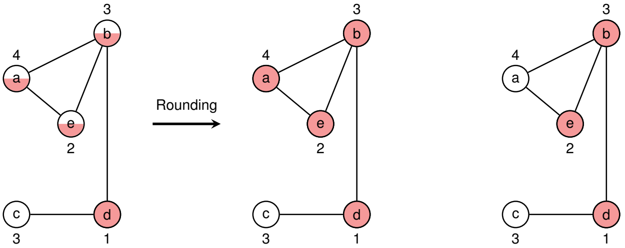

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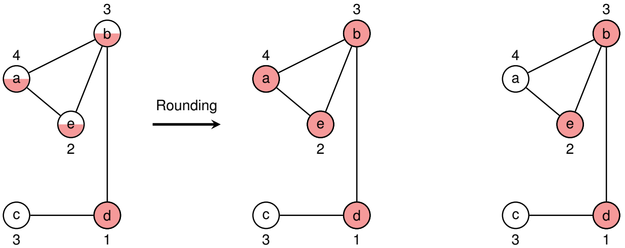

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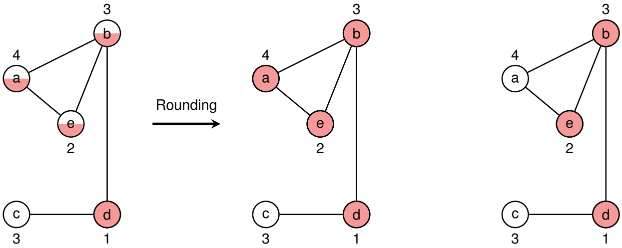

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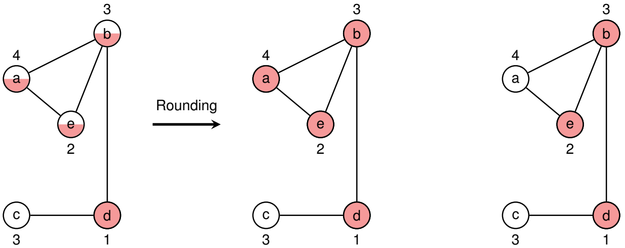

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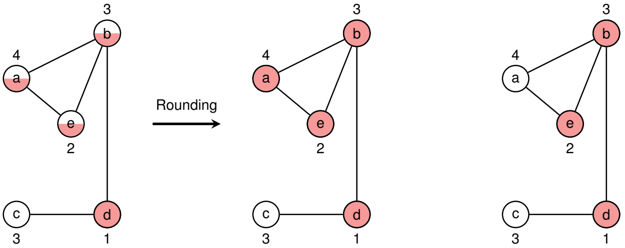

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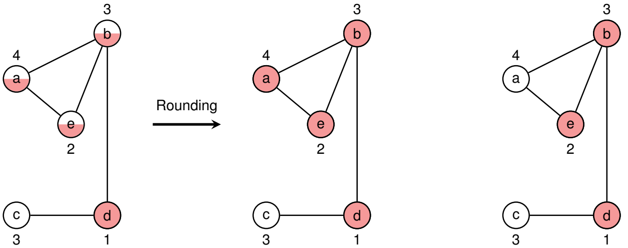

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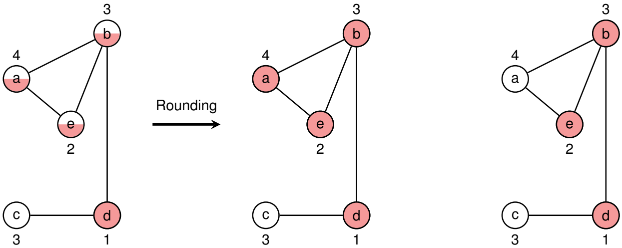

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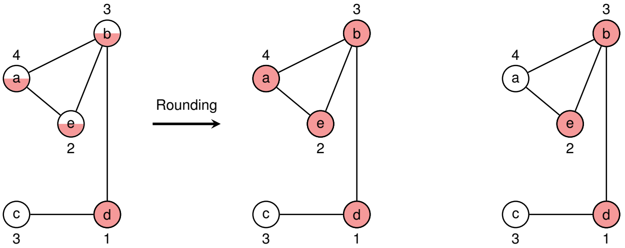

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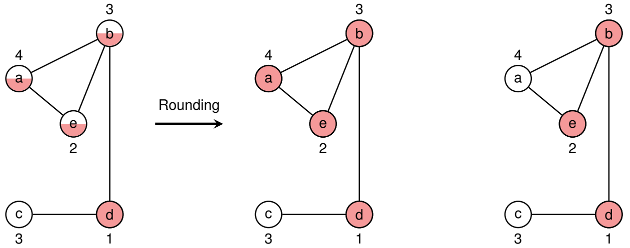

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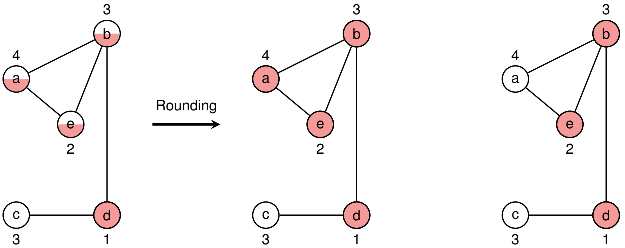

Approximation Ratio

Proof (Approximation Ratio is 2 and Correctness):

- Let C^* be an optimal solution to the minimum-weight vertex cover problem
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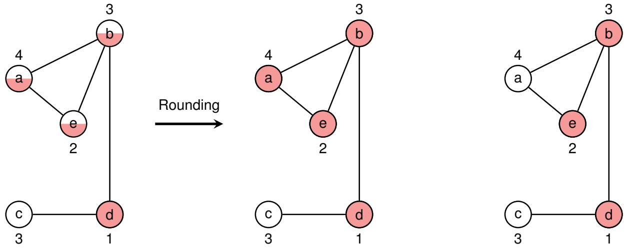

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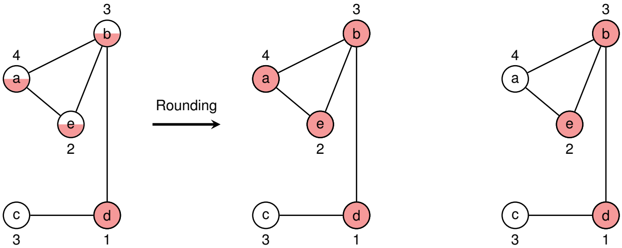

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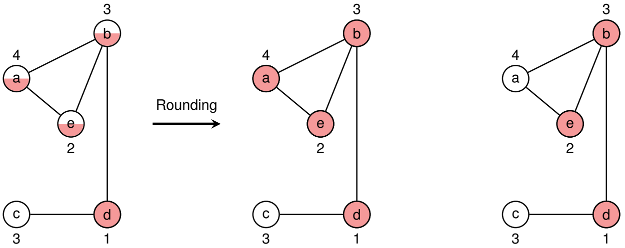

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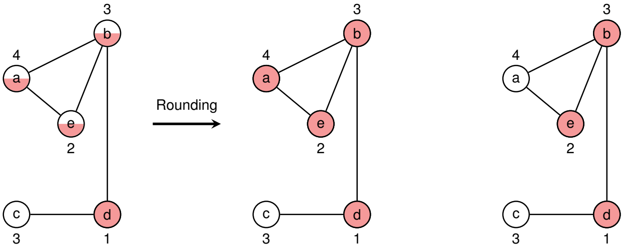

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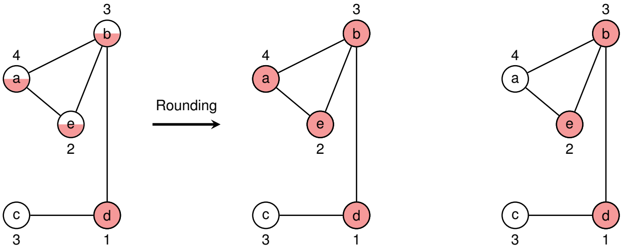

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

The Weighted Set-Covering Problem

Set Cover Problem

- Given: set X and a family of subsets $\mathcal{F}$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$
- Goal: Find a minimum-cost subset $\mathcal{C} \subseteq \mathcal{F}$

$$\text{s.t.} \qquad X = \bigcup_{S \in \mathcal{C}} S.$$

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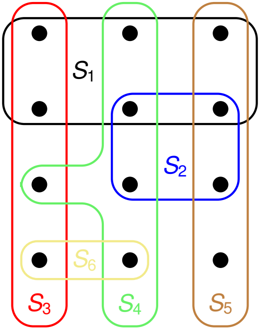
Sum over the costs of all sets in $\mathcal{C}$



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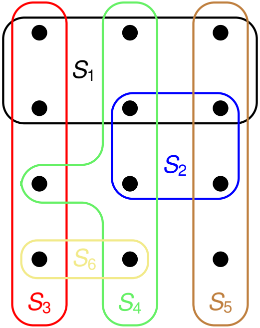

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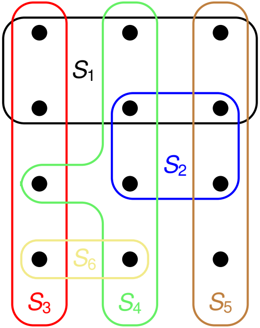

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Remarks:

- generalisation of the weighted vertex-cover problem
- models resource allocation problems

Setting up an Integer Program

Exercise: Try to formulate the integer program and linear program of the weighted SET-COVER problem (solution on next slide!)

Setting up an Integer Program

0-1 Integer Program

$$
\begin{array}{llcll}
\text{minimize} & \displaystyle\sum_{S \in \mathcal{F}} c(S) y(S) & & & \\
\text{subject to} & \displaystyle\sum_{S \in \mathcal{F}\colon\, x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\
 & y(S) & \in & \{0,1\} & \text{for each } S \in \mathcal{F}
\end{array}
$$

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Linear Program

$$
\begin{array}{llcll}
\text{minimize} & \sum_{S \in \mathcal{F}} c(S)y(S) & & & \\
\text{subject to} & \sum_{S \in \mathcal{F}:\, x \in S} y(S) & \geq & 1 & \text{for each } x \in X \\
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\end{array}
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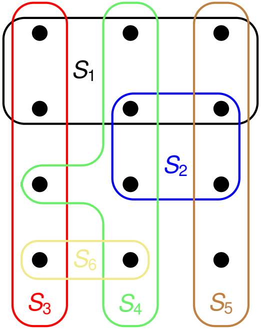
Back to the Example



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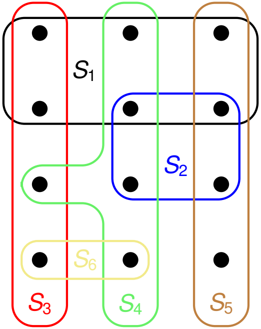

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$\overline{y}(.)$:	$1/2$	$1/2$	$1/2$	$1/2$	1	$1/2$

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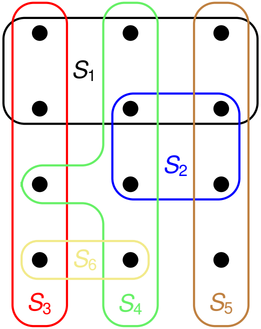

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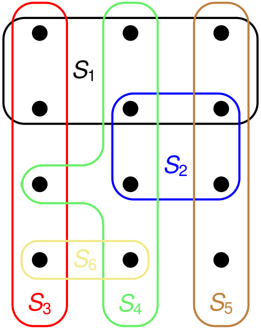

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The strategy employed for Vertex-Cover would take all 6 sets!

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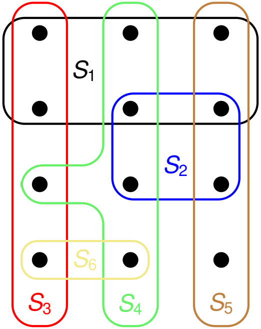

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Even worse: If all $\overline{y}$'s were below $1/2$, we would not even return a valid cover!

Randomised Rounding

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Idea: Interpret the $\overline{y}$-values as probabilities for picking the respective set.

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Randomised Rounding

- Let $\mathcal{C} \subseteq \mathcal{F}$ be a random set with each set S being included independently with probability $\overline{y}(S)$.
- More precisely, if $\overline{y}$ denotes the optimal solution of the LP, then we compute an integral solution y by:

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- Therefore, $\mathbf{E}\left[\, y(S) \,\right] = \overline{y}(S)$.

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$$\mathbf{E}\left[\, c(\mathcal{C}) \,\right] = \sum_{S \in \mathcal{F}} c(S) \cdot \overline{y}(S)$$

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- The probability that an element $x \in X$ is covered satisfies

$$\mathbf{P}\left[x \in \bigcup_{S \in \mathcal{C}} S \right] \geq 1 - \frac{1}{e}.$$

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$$\begin{aligned}\mathbf{E}[\, c(\mathcal{C}) \,] &= \mathbf{E}\left[\sum_{S \in \mathcal{C}} c(S) \right] = \mathbf{E}\left[\sum_{S \in \mathcal{F}} \mathbf{1}_{S \in \mathcal{C}} \cdot c(S) \right] \\ &= \sum_{S \in \mathcal{F}} \mathbf{P}[\, S \in \mathcal{C} \,] \cdot c(S) = \sum_{S \in \mathcal{F}} \overline{y}(S) \cdot c(S).\end{aligned}$$

- **Step 2**: The probability for an element to be (not) covered

$$\mathbf{P}[\, x \notin \cup_{S \in \mathcal{C}} S \,]$$

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compute ȳ, an optimal solution to the linear program
C = ∅
repeat 2 ln n times
    for each S ∈ F
        let C = C ∪ {S} with probability ȳ(S)
return C
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repeat $2 \ln n$ times
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return $\mathcal{C}$

clearly runs in polynomial-time!

Analysis of WEIGHTED SET COVER-LP

Theorem

- With probability at least $1 - \frac{1}{n}$, the returned set $\mathcal{C}$ is a valid cover of X.
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Typical Approach for Designing Approximation Algorithms based on LPs

Outline

Randomised Approximation

MAX-3-CNF

Weighted Vertex Cover

Weighted Set Cover

MAX-CNF

MAX-CNF

Recall:

MAX-3-CNF Satisfiability

- Given: 3-CNF formula, e.g.: $(x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5}) \wedge \cdots$
- Goal: Find an assignment of the variables that satisfies as many clauses as possible.

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Why study this generalised problem?

- Allowing arbitrary clause lengths makes the problem more interesting (we will see that simply guessing is not the best!)
- a nice concluding example where we can practice previously learned approaches

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Assign each variable true or false uniformly and independently at random.

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$$\mathbf{E}[\,Y\,] = \mathbf{E}\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} \mathbf{E}[\,Y_i\,] \geq \sum_{i=1}^{m} \frac{1}{2} = \frac{1}{2} \cdot m. \qquad \square$$

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First solve a linear program and use fractional values for a **biased** coin flip.

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\text{maximize} & \displaystyle\sum_{i=1}^{m} z_i & & & \\
\text{subject to} & \displaystyle\sum_{j \in C_i^+} y_j + \sum_{j \in C_i^-} (1 - y_j) & \geq & z_i & \text{for each } i = 1, 2, \ldots, m \\
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- In the corresponding LP each $\in \{0, 1\}$ is replaced by $\in [0, 1]$
- Let $(\overline{y}, \overline{z})$ be the optimal solution of the LP
- Obtain an integer solution y through randomised rounding of $\overline{y}$

Analysis of Randomised Rounding

Lemma

For any clause i of length ℓ,

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$$\Rightarrow \quad \mathbf{P}\left[\,\text{clause } i \text{ is satisfied}\,\right] = 1 - \prod_{j=1}^{\ell} \mathbf{P}\left[\; y_j \text{ is false } \;\right] = 1 - \prod_{j=1}^{\ell} (1 - \overline{y}_j)$$

Arithmetic vs. geometric mean:

$$\frac{a_1 + \ldots + a_k}{k} \geq \sqrt[k]{a_1 \times \ldots \times a_k}.$$

$$\geq 1 - \left(\frac{\sum_{j=1}^{\ell}(1 - \overline{y}_j)}{\ell}\right)^{\ell}$$

$$= 1 - \left(1 - \frac{\sum_{j=1}^{\ell} \overline{y}_j}{\ell}\right)^{\ell} \geq 1 - \left(1 - \frac{\overline{z}_i}{\ell}\right)^{\ell}.$$

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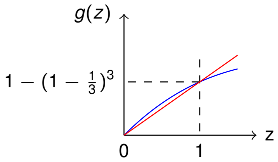

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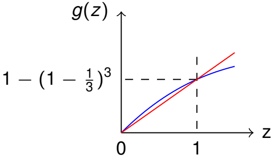

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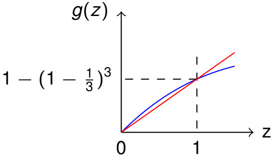

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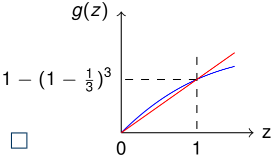

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LP solution at least as good as optimum

Approach 3: Hybrid Algorithm

Summary

- Approach 1 (Guessing) achieves better guarantee on longer clauses
- Approach 2 (Rounding) achieves better guarantee on shorter clauses

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HYBRID-MAX-CNF(φ, n, m)

1: Let $b \in \{0, 1\}$ be the flip of a fair coin
2: **If** $b = 0$ **then** perform random guessing
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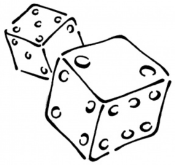

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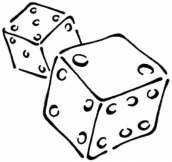

Algorithm sets each variable x_i to TRUE with prob. $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \overline{y}_i$.
Note, however, that variables are **not** independently assigned!

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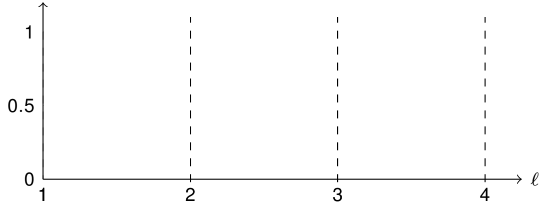

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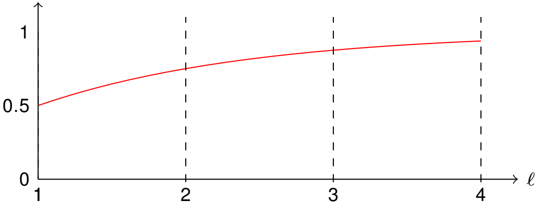

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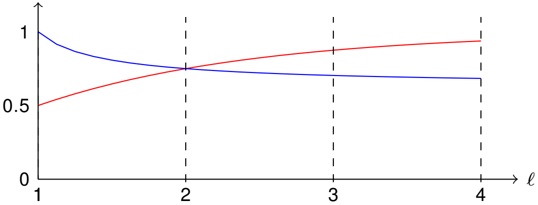

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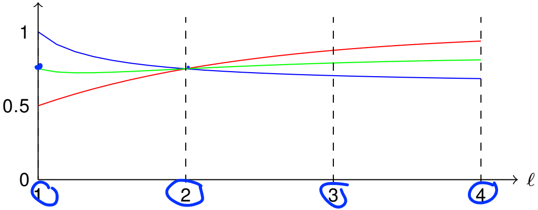

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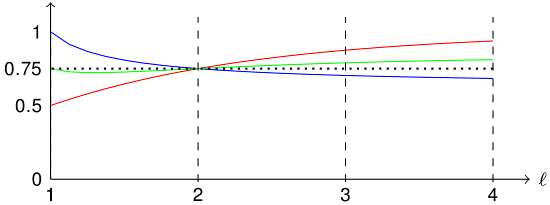

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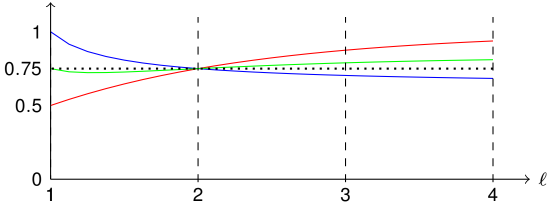
- $\Rightarrow$ HYBRID-MAX-CNF(φ, n, m) satisfies it with prob. at least $3/4 \cdot \overline{z}_i$ □



MAX-CNF Conclusion

Summary

- Since $\alpha_2 = \beta_2 = 3/4$, we cannot achieve a better approximation ratio than $4/3$ by combining Algorithm 1 & 2 in a different way
- The $4/3$-approximation algorithm can be easily derandomised
  - Idea: use the conditional expectation trick for both Algorithm 1 & 2 and output the better solution
- The $4/3$-approximation algorithm applies unchanged to a weighted version of MAX-CNF, where each clause has a non-negative weight
- Even MAX-2-CNF (every clause has length 2) is NP-hard!

Exercise (easy): Consider a minimisation problem, where x is the optimal cost of the LP relaxation, y is the optimal cost of the IP and z is the solution obtained by rounding the LP solution. Which of the following statements are true?

1. $z \leq x \leq y$,
2. $x \leq y \leq z$,
3. $y \leq x \leq z$,
4. $y \leq z \leq x$.