Randomised Algorithms

Lecture 5: Random Walks, Hitting Times and Application to 2-SAT

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Lent 2022

UNIVERSITY OF CAMBRIDGE

Outline

Random Walks on Graphs

A Simple Random Walk (SRW) on a graph G is a Markov chain on $V(G)$ with

$$P(u,v) = \begin{cases} \frac{1}{\deg(u)} & \text{if } \{u,v\} \in E, \\ 0 & \text{if } \{u,v\} \notin E. \end{cases}, \qquad \text{and} \qquad \pi(u) = \frac{\deg(u)}{2|E|}$$

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Recall: $h(u,v) = \mathbf{E}_u[\min\{t \geq 1 \colon X_t = v\}]$ is the hitting time of v from u.

Lazy Random Walks and Periodicity

The Lazy Random Walk (LRW) on G given by $\widetilde{P} = (P + I)/2$,

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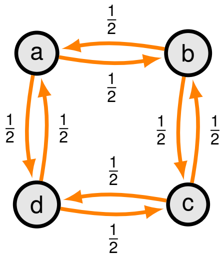

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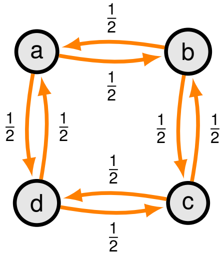

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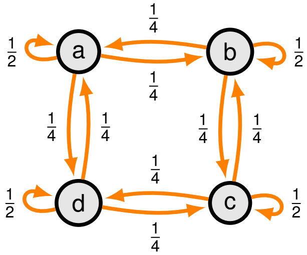

LRW on C_4, *Aperiodic*

Application: Cover Time and Undirected Connectivity

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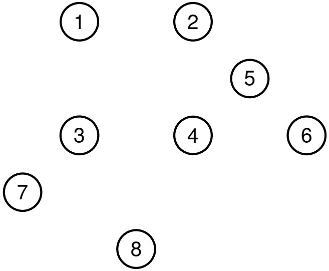

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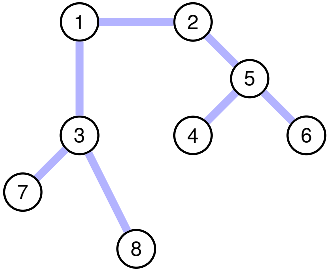

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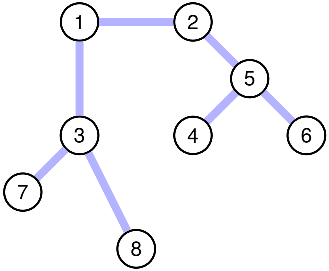

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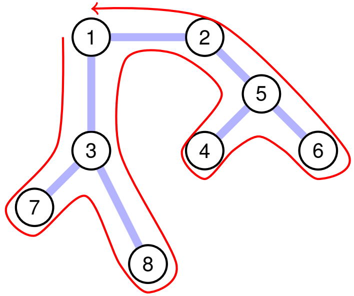

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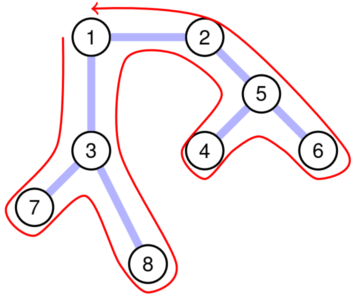

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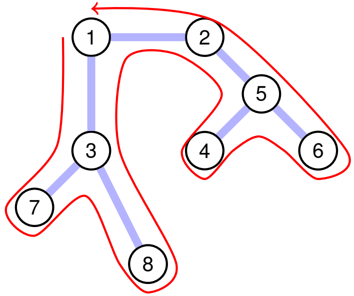

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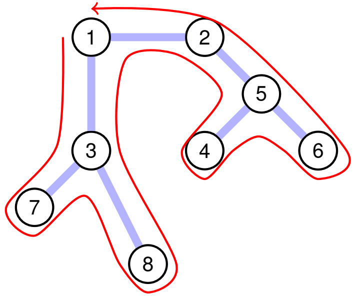

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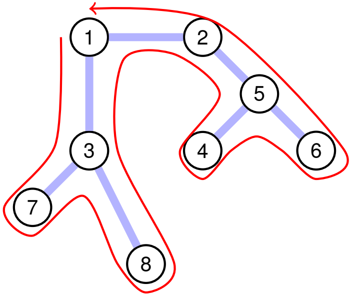

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$\Rightarrow$ solves the USTCON problem with $O(\log n)$ space (see Complexity Theory course)

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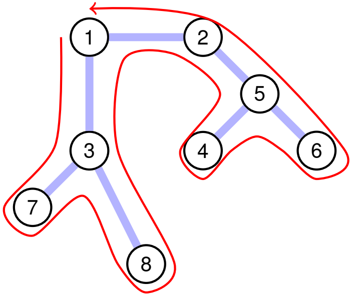

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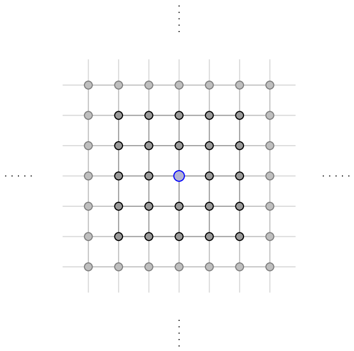

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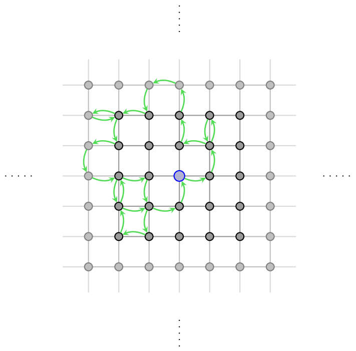

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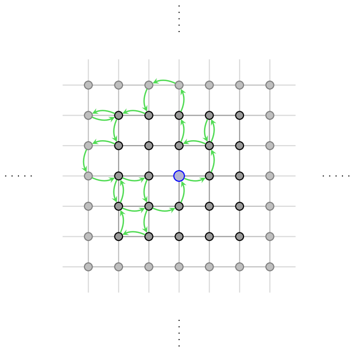

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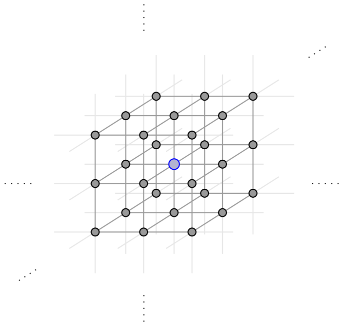

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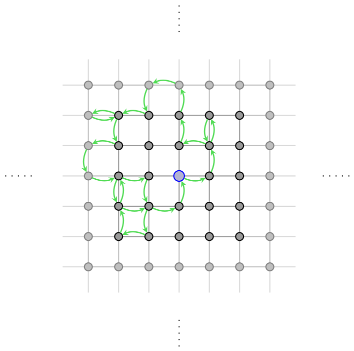

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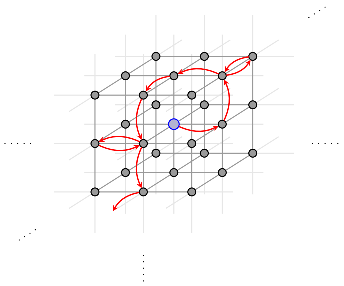

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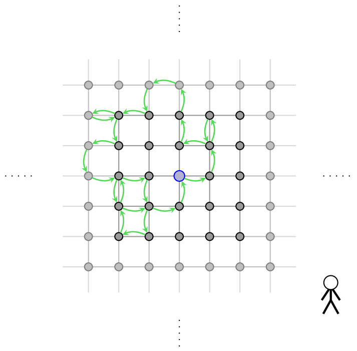

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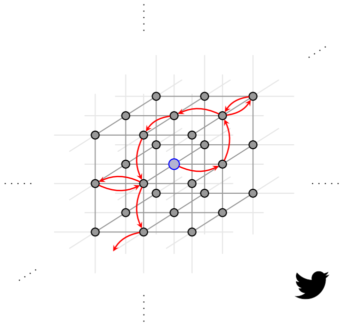

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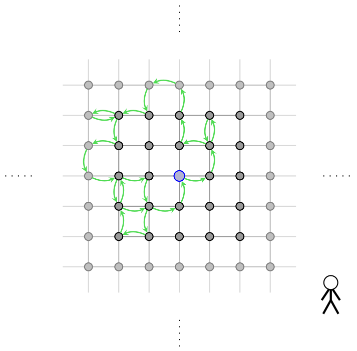

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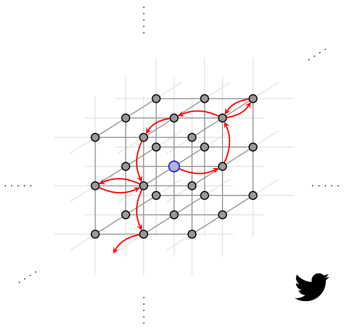

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But for any regular (finite) graph, the expected return time to u is $1/\pi(u) = n$

SRW Random Walk on Two-Dimensional Grids: Animation

Random Walk on a Path (1/2)

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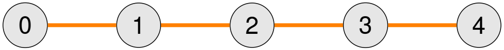

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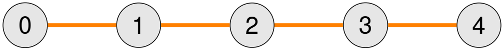

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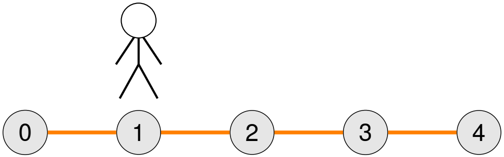

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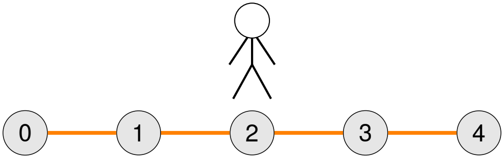

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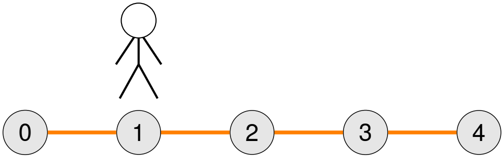

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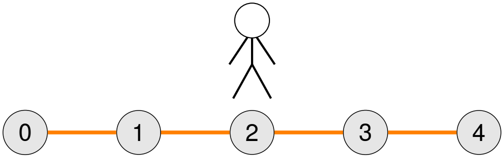

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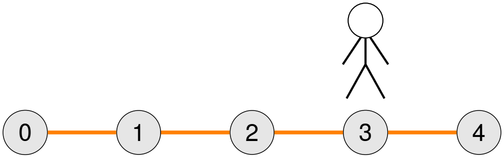

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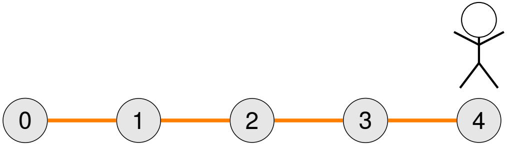

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Outline

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RANDOMISED2-SAT (Input: a 2-SAT-Formula)

2-SAT

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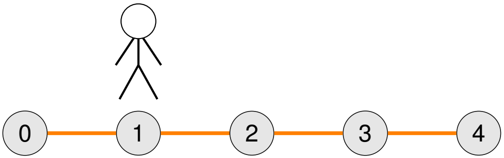

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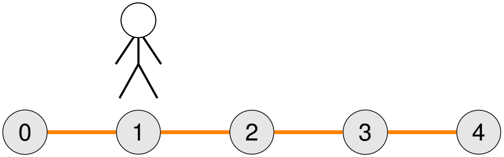

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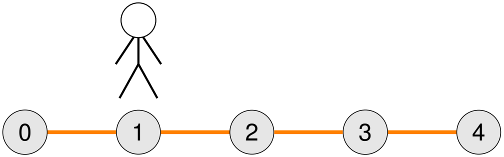

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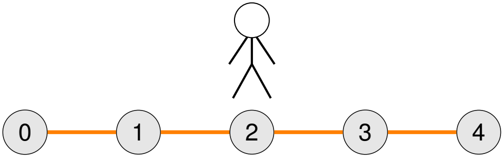

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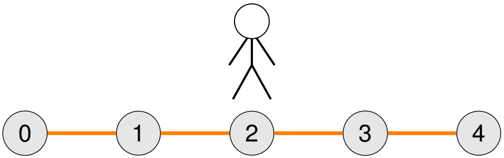

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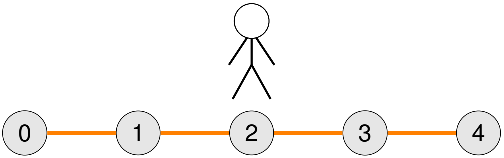

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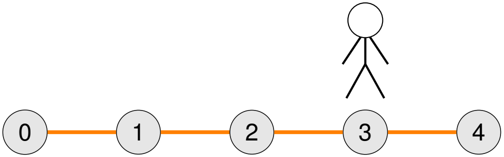

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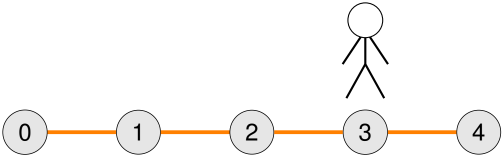

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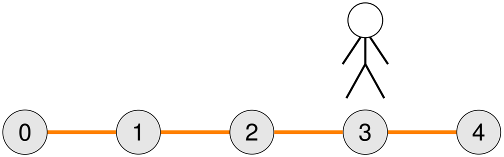

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- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

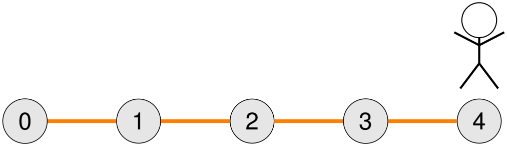
Example 1 :

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee \overline{x_3}) \wedge (x_4 \vee \overline{x_1})$$

T F F T T T T T T F

$$\alpha = (\mathtt{T}, \mathtt{T}, \mathtt{F}, \mathtt{T}).$$

t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	T	F	F
2	T	T	F	F
3	T	T	F	T



2-SAT

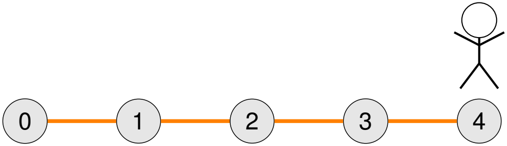
```
RANDOMISED2-SAT (Input: a 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
3:     Pick an arbitrary unsatisfied clause
4:     Choose a random literal and switch its value
5:     If formula is satisfied then return "Satisfiable"
6: return "Unsatisfiable"
```

- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |$variable values shared by A_i and $\alpha|$.

Example 1 : Solution Found

$$\underset{\mathtt{T}}{(x_1} \vee \underset{\mathtt{F}}{\overline{x_2})} \wedge \underset{\mathtt{F}}{(\overline{x_1}} \vee \underset{\mathtt{T}}{\overline{x_3})} \wedge \underset{\mathtt{T}}{(x_1} \vee \underset{\mathtt{T}}{x_2)} \wedge \underset{\mathtt{T}}{(x_4} \vee \underset{\mathtt{T}}{\overline{x_3})} \wedge \underset{\mathtt{T}}{(x_4} \vee \underset{\mathtt{F}}{\overline{x_1})}$$

$$\alpha = (\mathtt{T}, \mathtt{T}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	T	F	F
2	T	T	F	F
3	T	T	F	T

2-SAT

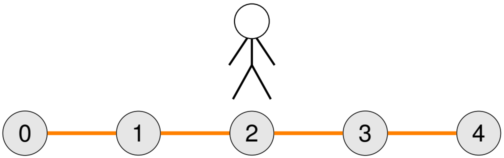
```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
3:     Pick an arbitrary unsatisfied clauses
4:     Choose a random literal and switch its value
5:     If formula is satisfied then return “Satisfiable”
6: return “Unsatisfiable”
```

- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

Example 2 :

$$\begin{array}{ccccccccccccccc} (x_1 & \vee & \overline{x_2}) & \wedge & (\overline{x_1} & \vee & \overline{x_3}) & \wedge & (x_1 & \vee & x_2) & \wedge & (x_4 & \vee & x_3) & \wedge & (x_4 & \vee & \overline{x_1}) \\ \mathtt{F} & & \mathtt{T} & & \mathtt{T} & & \mathtt{T} & & \mathtt{F} & & \mathtt{F} & & \mathtt{F} & & \mathtt{F} & & \mathtt{F} & & \mathtt{T} \end{array}$$

$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T})$.



t	x_1	x_2	x_3	x_4
0	F	F	F	F

2-SAT

RANDOMISED2-SAT (Input: A 2-SAT-Formula)

```
1: Start with an arbitrary truth assignment
2: **Repeat up to** $2n^2$ **times**
3:     Pick an arbitrary unsatisfied clauses
4:     Choose a random literal and switch its value
5:     **If** formula is satisfied **then return** “Satisfiable”
6: **return** “Unsatisfiable”
```

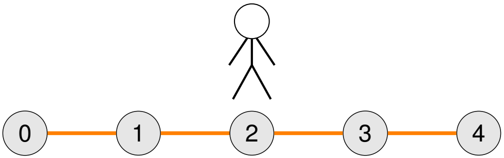
- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

Example 2 :

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F T T T F F F F F T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
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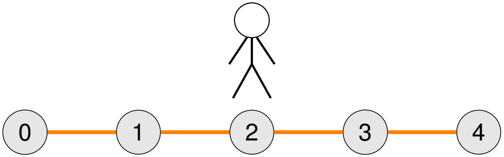
- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |$variable values shared by A_i and $\alpha|$.

Example 2 :

$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F T T T F F F F F T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
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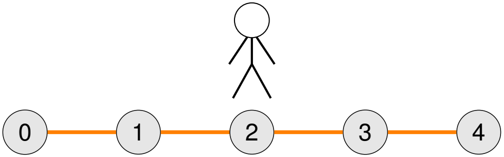
$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F T T T F F F F F T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F

2-SAT

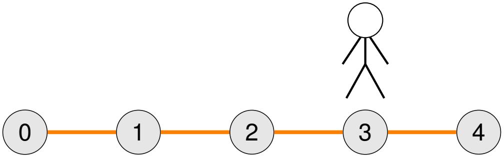
```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
3:     Pick an arbitrary unsatisfied clauses
4:     Choose a random literal and switch its value
5:     If formula is satisfied then return “Satisfiable”
6: return “Unsatisfiable”
```

- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |$variable values shared by A_i and $\alpha|$.

Example 2 :

$$\underset{\mathtt{F}}{(x_1} \vee \underset{\mathtt{T}}{\overline{x_2})} \wedge \underset{\mathtt{T}}{(\overline{x_1}} \vee \underset{\mathtt{T}}{\overline{x_3})} \wedge \underset{\mathtt{F}}{(x_1} \vee \underset{\mathtt{F}}{x_2)} \wedge \underset{\mathtt{T}}{(x_4} \vee \underset{\mathtt{F}}{x_3)} \wedge \underset{\mathtt{T}}{(x_4} \vee \underset{\mathtt{T}}{\overline{x_1})}$$

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: Repeat up to $2n^2$ times
3:     Pick an arbitrary unsatisfied clauses
4:     Choose a random literal and switch its value
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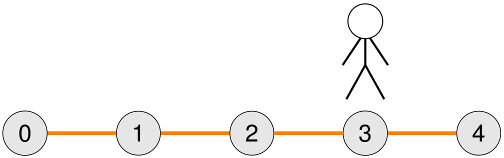
$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F T T T F F T F T T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
2: **Repeat up to** $2n^2$ **times**
3:     Pick an arbitrary unsatisfied clauses
4:     Choose a random literal and switch its value
5:     **If** formula is satisfied **then return** "Satisfiable"
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```

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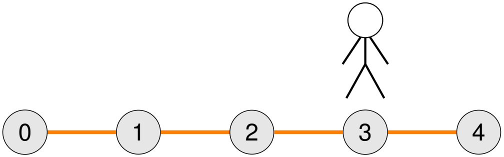
$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F T T T F F T F T T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
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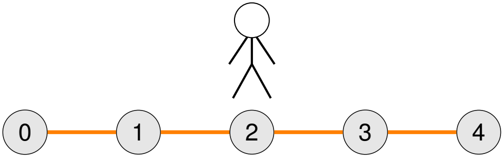
$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F F T T F T T F T T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T
2	F	T	F	T

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
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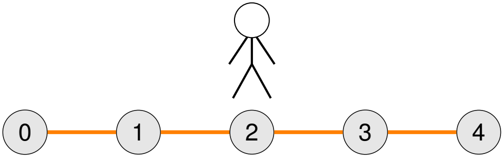
F F T T F T T F T T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T
2	F	T	F	T

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
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- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
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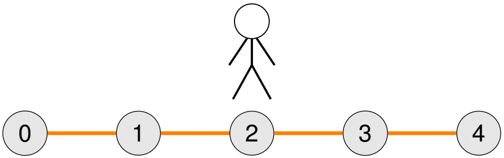
F F T T F T T F T T

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T
2	F	T	F	T

2-SAT

RANDOMISED2-SAT (Input: A 2-SAT-Formula)

```
1: Start with an arbitrary truth assignment
2: **Repeat up to** $2n^2$ **times**
3:     Pick an arbitrary unsatisfied clauses
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- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
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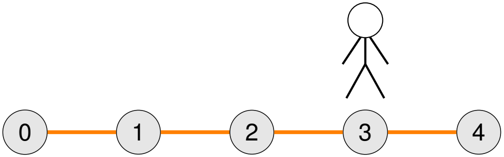
T F F T T T T F T F

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T
2	F	T	F	T
3	T	T	F	T

2-SAT

```
RANDOMISED2-SAT (Input: A 2-SAT-Formula)
1: Start with an arbitrary truth assignment
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4:     Choose a random literal and switch its value
5:     If formula is satisfied then return "Satisfiable"
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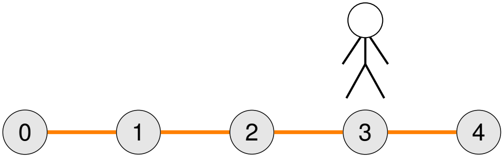
```

- Call each loop of (2) a step. Let A_i be the variable assignment at step i.
- Let α be any solution and $X_i = |\text{variable values shared by } A_i \text{ and } \alpha|$.

Example 2 : (Another) Solution Found

$$\underset{\mathtt{T}}{(x_1} \vee \underset{\mathtt{F}}{\overline{x_2})} \wedge \underset{\mathtt{F}}{(\overline{x_1}} \vee \underset{\mathtt{T}}{\overline{x_3})} \wedge \underset{\mathtt{T}}{(x_1} \vee \underset{\mathtt{T}}{x_2)} \wedge \underset{\mathtt{T}}{(x_4} \vee \underset{\mathtt{F}}{x_3)} \wedge \underset{\mathtt{T}}{(x_4} \vee \underset{\mathtt{F}}{\overline{x_1})}$$

$$\alpha = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F
1	F	F	F	T
2	F	T	F	T
3	T	T	F	T

2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED2-SAT outputs a valid solution is at most n^2.

2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED2-SAT outputs a valid solution is at most n^2.

Proof: Fix any solution α, then for any $i \geq 0$ and $1 \leq k \leq n-1$,

2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED2-SAT outputs a valid solution is at most n^2.

Proof: Fix any solution α, then for any $i \geq 0$ and $1 \leq k \leq n-1$,

(i) $\mathbf{P}[X_{i+1} = 1 \mid X_i = 0] = 1$

2-SAT and the SRW on the Path

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(i) $\mathbf{P}[\, X_{i+1} = 1 \mid X_i = 0 \,] = 1$

(ii) $\mathbf{P}[\, X_{i+1} = k+1 \mid X_i = k \,] \geq 1/2$

2-SAT and the SRW on the Path

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(iii) $\mathbf{P}[\, X_{i+1} = k-1 \mid X_i = k \,] \leq 1/2$.

2-SAT and the SRW on the Path

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(iii) $\mathbf{P}[X_{i+1} = k-1 \mid X_i = k] \leq 1/2$.

Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

2-SAT and the SRW on the Path

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Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).

2-SAT and the SRW on the Path

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Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).

The stochastic process X_i is complicated to describe in full; however by $(i) - (iii)$ we can **bound** it by Y_i (SRW on the n-path from 0).

2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED2-SAT outputs a valid solution is at most n^2.

Proof: Fix any solution α, then for any $i \geq 0$ and $1 \leq k \leq n-1$,

(i) $\mathbf{P}[X_{i+1} = 1 \mid X_i = 0] = 1$

(ii) $\mathbf{P}[X_{i+1} = k+1 \mid X_i = k] \geq 1/2$

(iii) $\mathbf{P}[X_{i+1} = k-1 \mid X_i = k] \leq 1/2$.

Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).

The stochastic process X_i is complicated to describe in full; however by $(i) - (iii)$ we can **bound** it by Y_i (SRW on the n-path from 0). This gives

$$\mathbf{E}[\text{time to find sol}] \leq \mathbf{E}_0[\min\{t : X_t = n\}] \leq \mathbf{E}_0[\min\{t : Y_t = n\}] = h(0,n) = n^2.$$

□

2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED2-SAT outputs a valid solution is at most n^2.

Proof: Fix any solution α, then for any $i \geq 0$ and $1 \leq k \leq n-1$,

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Notice that if $X_i = n$ then $A_i = \alpha$ thus solution found (may find another first).

Assume (pessimistically) that $X_0 = 0$ (none of our initial guesses is right).

The stochastic process X_i is complicated to describe in full; however by $(i) - (iii)$ we can **bound** it by Y_i (SRW on the n-path from 0). This gives

$$\mathbf{E}[\text{time to find sol}] \leq \mathbf{E}_0[\min\{t : X_t = n\}] \leq \mathbf{E}_0[\min\{t : Y_t = n\}] = h(0,n) = n^2.$$

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Running for $2n^2$ time and using Markov's inequality yields:

2-SAT and the SRW on the Path

Expected iterations of (2) in RANDOMISED2-SAT

If the formula is satisfiable, then the expected number of steps before RANDOMISED2-SAT outputs a valid solution is at most n^2.

Proof: Fix any solution α, then for any $i \geq 0$ and $1 \leq k \leq n-1$,

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Proposition

Provided a solution exists, RANDOMISED2-SAT will return a valid solution in $O(n^2)$ time with probability at least $1/2$.

Boosting Success Probabilities

Boosting Lemma

Suppose a randomised algorithm succeeds with probability (at least) p. Then for any $C \geq 1$, $\lceil \frac{C}{p} \cdot \log n \rceil$ repetitions are sufficient to succeed (in at least one repetition) with probability at least $1 - n^{-C}$.

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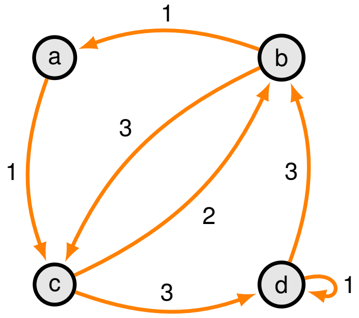
RANDOMISED2-SAT

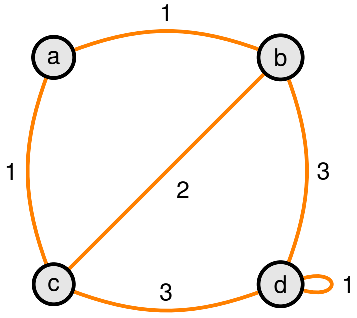
There is a $O(n^2 \log n)$-time algorithm for 2-SAT which succeeds w.h.p.

Outline

Random Walks on Weighted Graphs

An (edge) weighted graph $G = (V, E, w)$ where $w : E \to \mathbb{R}_+$ on the edges.





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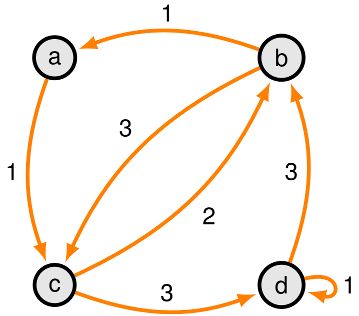
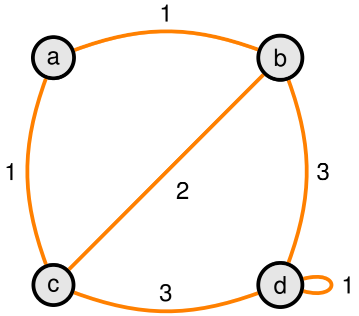


Directed



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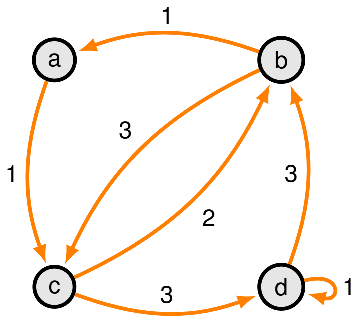

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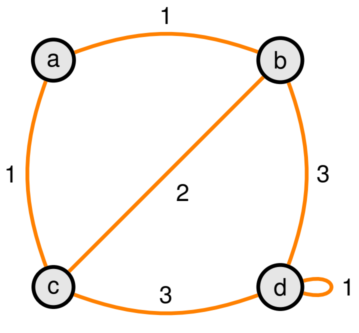

Undirected

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- Reversible Markov Chains are equivalent to random walks on weighted **undirected** graphs.
- A reversible Markov Chain identified with the (undirected) weighted graph $G = (V, E, w)$ has stationary distribution given by

$$\pi(i) = \frac{\sum_{j:\{i,j\}\in E} w(i,j)}{2\sum_{\{x,y\}\in E} w(x,y)}$$