Randomised Algorithms

Lecture 2-3: Concentration Inequalities

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UNIVERSITY OF CAMBRIDGE

Outline

Concentration Inequalities

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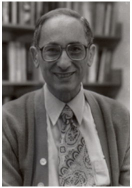
- This is very useful in randomised algorithms as it ensures an **almost** deterministic behaviour
- It gives us the best of two worlds:
  1. **Randomised Algorithms:** Easy to Design and Implement
  2. **Deterministic Algorithms:** They do what they claim

Chernoff Bounds: A Tool for Concentration

- Chernoffs bounds are "strong" bounds on the tail probabilities of sums of independent random variables
- random variables can be discrete (or continuous)
- usually these bounds decrease exponentially as opposed to a polynomial decrease in Markov's or Chebyshev's inequality (see example)



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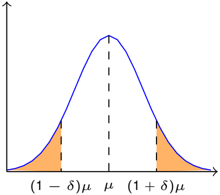

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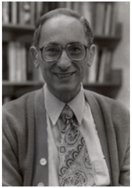

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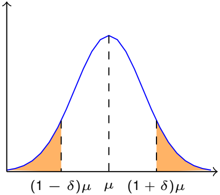

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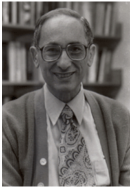
- have found various applications in:
  - Randomised Algorithms
  - Statistics
  - Random Projections and Dimensionality Reduction
  - Learning Theory (e.g., PAC-learning)

  ⋮



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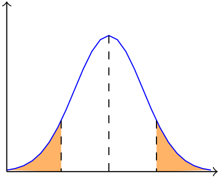

Recap: Markov and Chebyshev

Markov's Inequality

If X is a non-negative random variable, then for any $a > 0$,

$$\mathbf{P}[X \geq a] \leq \mathbf{E}[X]/a.$$

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Chebyshev's inequality (or Markov) can be obtained by chosing $f(X) := (X - \mu)^2$ (or $f(X) := X$, respectively).

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We can consider the first, second, third and more moments! That is the basic idea behind the Chernoff Bounds

Our First Chernoff Bound

Chernoff Bounds (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}. \qquad (\star)$$

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This implies that for any $t > \mu$,

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What about a concrete value of n, say $n = 100$?

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Chernoff bound yields a much better result (but needs independence!)

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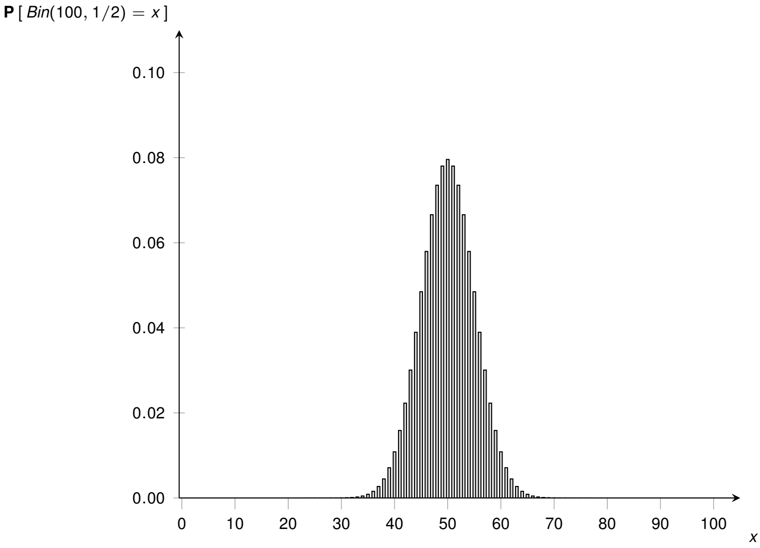

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3. Optimise value of λ to obtain best tail bound

Chernoff Bound: Proof

Chernoff Bound (General Form, Upper Tail)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^{n} p_i$. Then, for any $\delta > 0$ it holds that

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5. Choose $\lambda = \log(1+\delta) > 0$ to get the result.

Chernoff Bounds: Lower Tails

We can also use Chernoff Bounds to show a random variable is **not too small** compared to its mean:

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Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[X \leq (1-\delta)\mu] \leq \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu},$$

and thus, by substitution, for any $t < \mu$,

$$\mathbf{P}[X \leq t] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^{t}.$$

Exercise on Supervision Sheet

Hint: multiply both sides by -1 and repeat the proof of the Chernoff Bound

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- For all $t > 0$,

$$\mathbf{P}[X \geq \mathbf{E}[X] + t] \leq e^{-2t^2/n}$$

$$\mathbf{P}[X \leq \mathbf{E}[X] - t] \leq e^{-2t^2/n}$$

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Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum_{i=1}^{n} p_i$. Then,

- For all $t > 0$,

$$\mathbf{P}[X \geq \mathbf{E}[X] + t] \leq e^{-2t^2/n}$$

$$\mathbf{P}[X \leq \mathbf{E}[X] - t] \leq e^{-2t^2/n}$$

- For $0 < \delta < 1$,

$$\mathbf{P}[X \geq (1+\delta)\mathbf{E}[X]] \leq \exp\left(-\frac{\delta^2 \mathbf{E}[X]}{3}\right)$$

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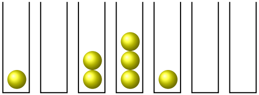
All upper tail bounds hold even under a relaxed independence assumption: For all $1 \leq i \leq n$ and $x_1, x_2, \ldots, x_{i-1} \in \{0, 1\}$,

$$\mathbf{P}[X_i = 1 \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq p_i.$$

Outline

Balls into Bins



Balls into Bins Model

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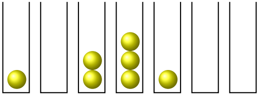
Balls into Bins



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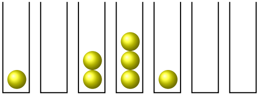

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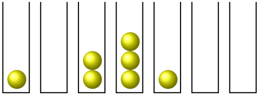

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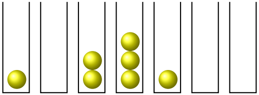

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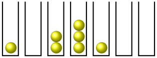
Exercise: Think about the relation between the Balls into Bins Model and the Coupon Collector Problem.

Balls into Bins: Bounding the Maximum Load (1/4)



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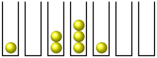

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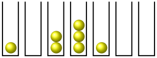

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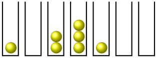

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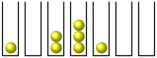

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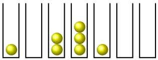

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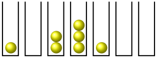

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here we could have used the "nicer" bounds as well!

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whp stands for *with high probability*:
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- By pigeonhole principle, the max loaded bin receives at least $2 \log n$ balls. Hence our bound is pretty sharp.

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Balls into Bins: Bounding the Maximum Load (4/4)

We just proved that

$$\mathbf{P}[X \geq 4 \log n / \log \log n] \leq n^{-2},$$

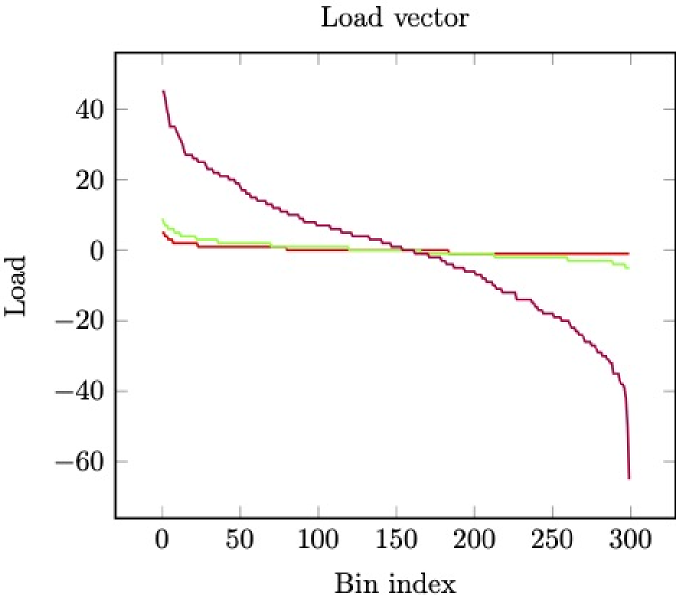
thus by the Union Bound, no bin receives more than $\Omega(\log n / \log \log n)$ balls with probability at least $1 - 1/n$. □

Simulations

- We plot the load configuration for $m \in \{n, n \log n, n^2\}$
- We consider $n \in \{300, 1000, 100000\}$
- In plots, we take the normalised load, that is, actual bin load minus average load

Acknowledgements: experiments and plots created by Dimitris Los

Balls-into-Bins Plot (1/3)



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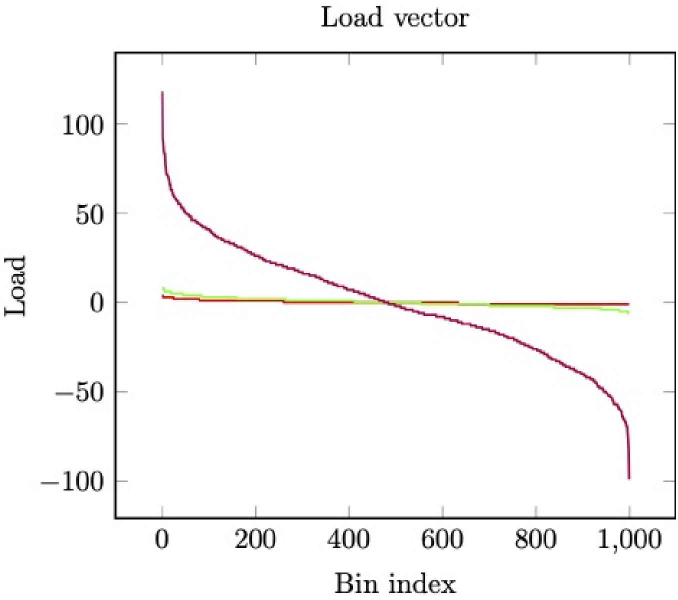
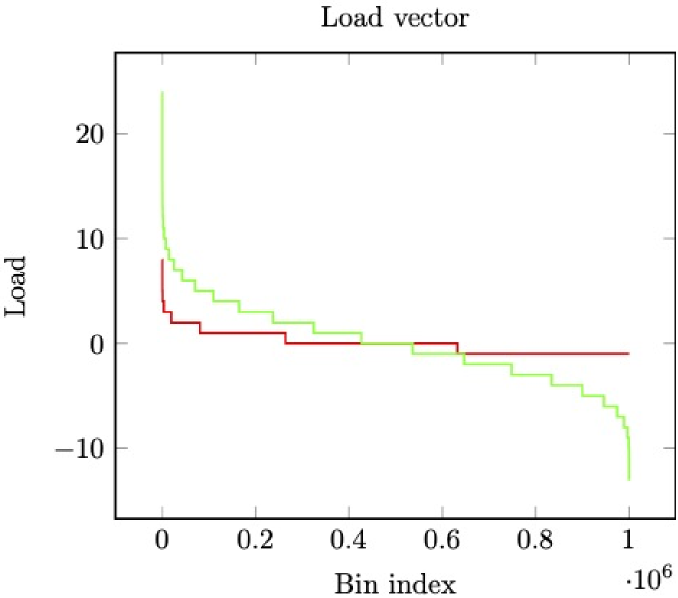


Balls-into-Bins Plot (3/3) (only $m \in \{n, n \log n\}$)



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This is called the **power of two choices**: It is a common technique to improve the performance of randomised algorithms.

Outline

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```
QUICKSORT (Input A[1], A[2], ..., A[n])
Pick an element from the array, the so-called pivot
If |A| = 0 or |A| = 1 then
    return A
else
    Create two subarrays A1 and A2 (without the pivot) such that:
        A1 contains the elements that are smaller than the pivot
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QuickSort

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Pick an element from the array, the so-called pivot
If $|A| = 0$ or $|A| = 1$ **then**
    return A
else
    Create two subarrays A_1 and A_2 (without the pivot) such that:
        A_1 contains the elements that are smaller than the pivot
        A_2 contains the elements that are greater (or equal) than the pivot
    QuickSort(A_1)
    QuickSort(A_2)
    return A

- Example: Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$ with $A[7] = 6$ as pivot.
  $\Rightarrow A_1 = (2, 1, 5, 3, 4)$ and $A_2 = (8, 9, 7)$
- Worst-Case Complexity (number of comparisons) is $\Theta(n^2)$,
  while Average-Case Complexity is $O(n \log n)$.

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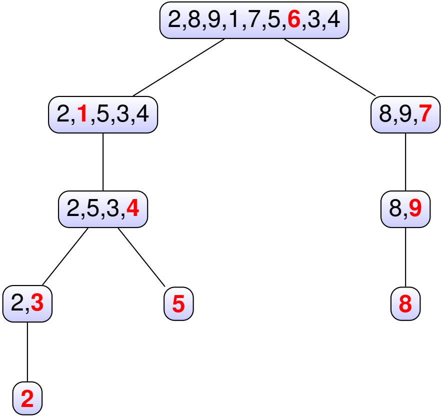
We will now give a proof of this “well-known” result!

QuickSort: How to Count Comparisons



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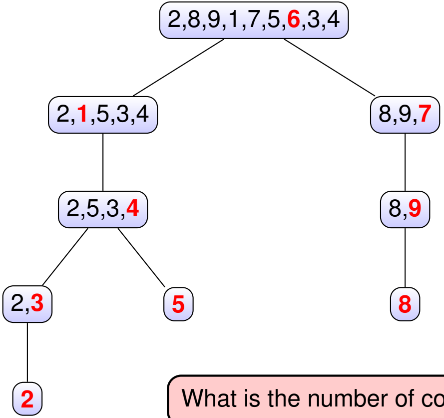

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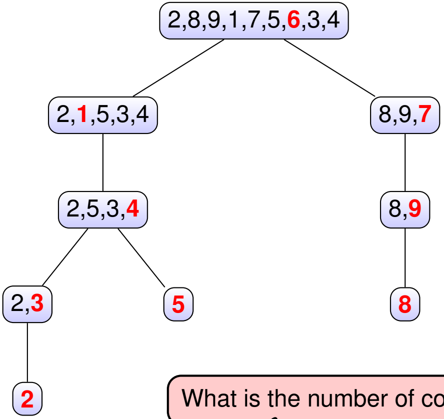

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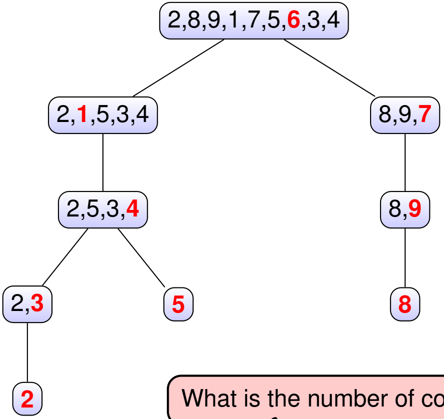

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Note that the number of comparison by QUICKSORT is equivalent to the sum of the height of all nodes in the tree (why?). In this case:

$$0 + 1 + 1 + 2 + 2 + 3 + 3 + 3 + 4 = 19.$$

Randomised QuickSort: Analysis (1/4)

How to pick a good pivot? We don't, **just pick one at random.**

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4. Actually, we will prove sth slightly stronger:

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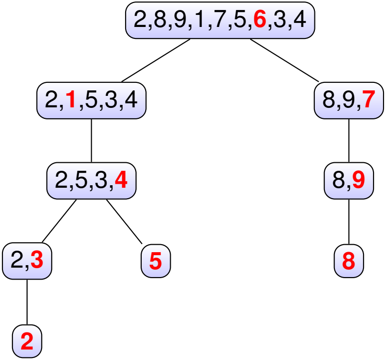

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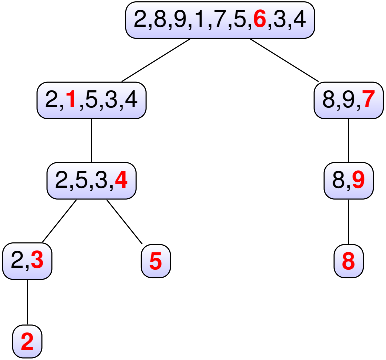

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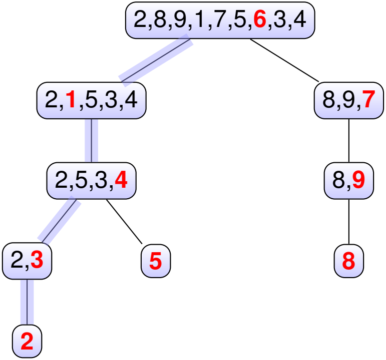

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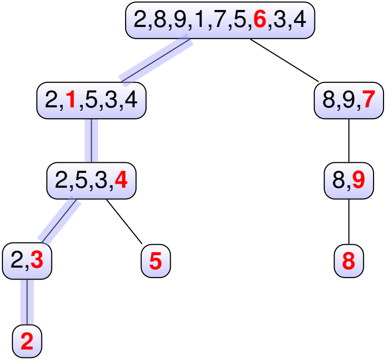

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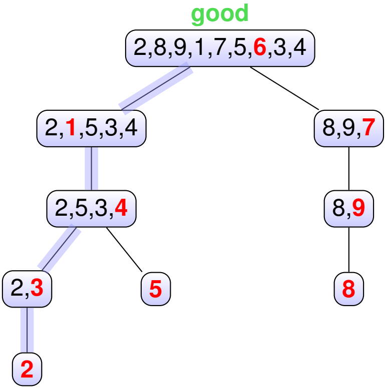

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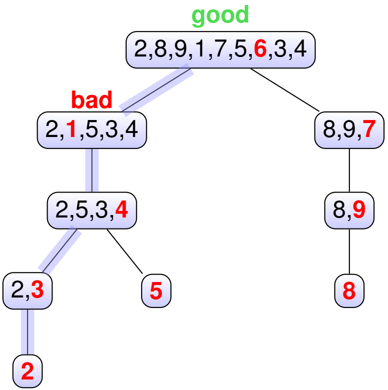

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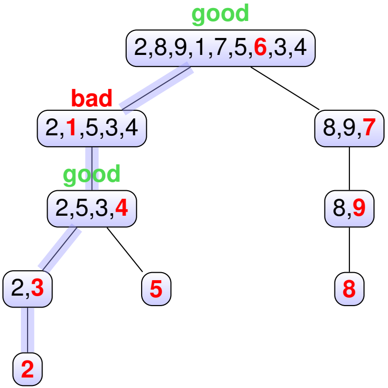

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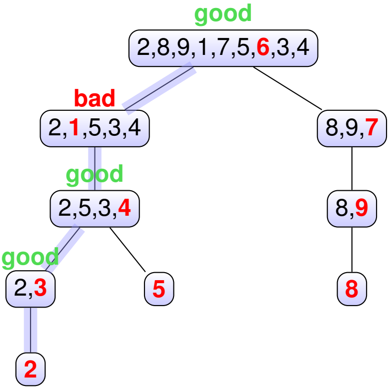

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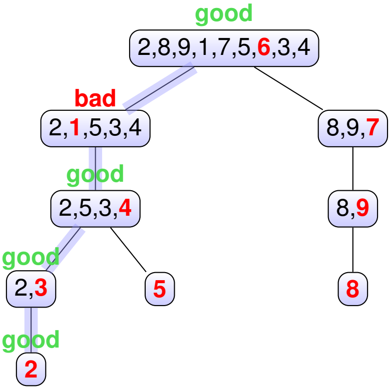

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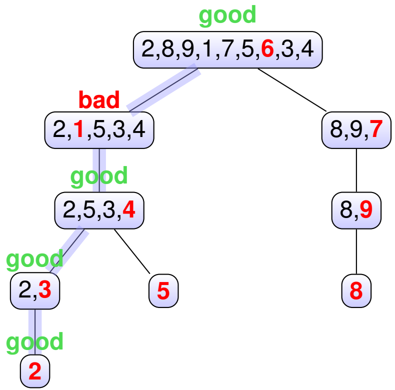

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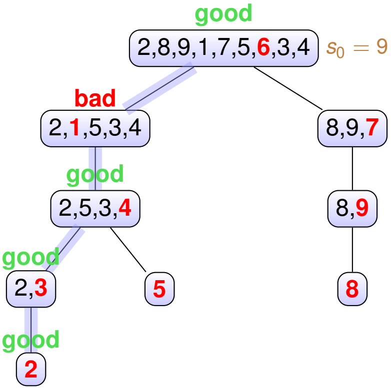

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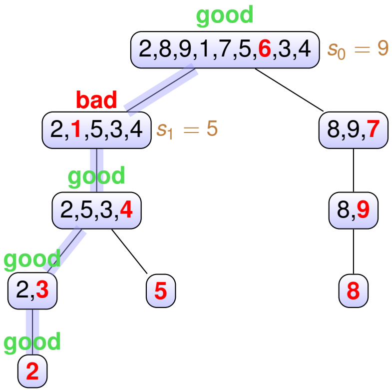

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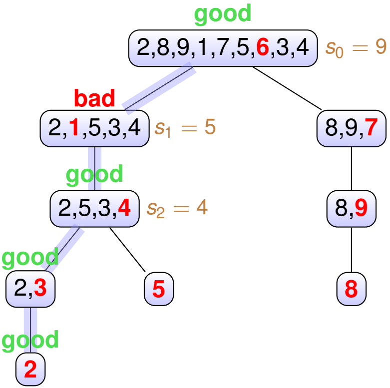

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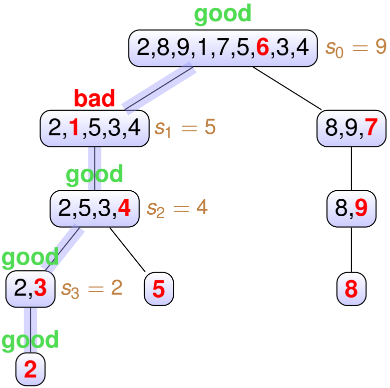

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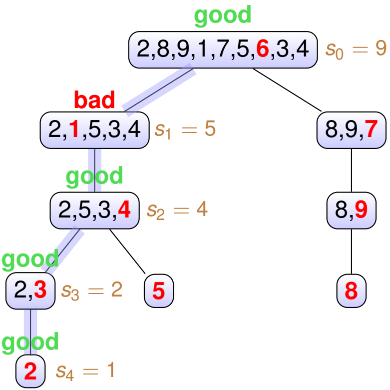

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Let us now upper bound the probability that this "bad event" happens!

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- $\mathbf{P}\left[X_j = 1 \mid X_0 = x_0, \ldots, X_{j-1} = x_{j-1} \right] \leq \frac{2}{3}$

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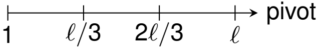

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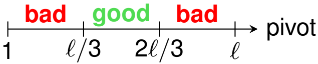

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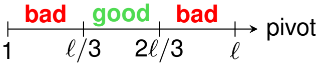

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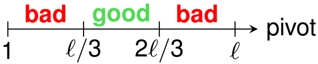

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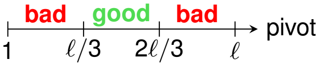

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Answer: We can then simply define X_j as the result of an independent coin flip with probability $2/3$.

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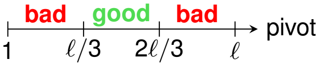

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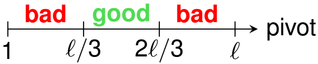

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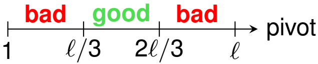

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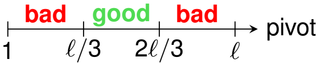

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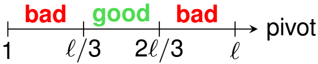

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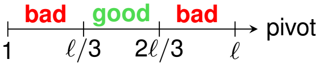

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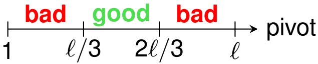

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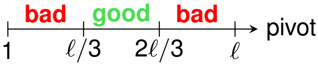

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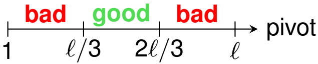

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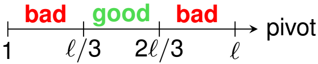

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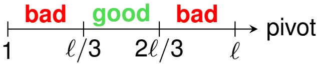

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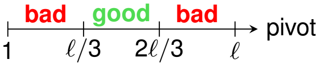

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> Supervision Exercise: Our upper bound of $O(n \log n)$ whp also immediately implies a $O(n \log n)$ bound on the expected number of comparisons!

- It is possible to deterministically find the best pivot element that divides the array into two subarrays of the same size.
- The latter requires to compute the median of the array in linear time, which is not easy...
- The presented randomised algorithm for QuickSort is much easier to implement!

Outline

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We omit the proof of this lemma!

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Let $X_1, \ldots, X_n$ be independent random variable with mean μ_i such that $a_i \leq X_i \leq b_i$. Let $X = X_1 + \ldots + X_n$, and let $\mu = \mathbf{E}[X] = \sum_{i=1}^n \mu_i$. Then for any $t > 0$

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Proof Outline (skipped):

- Let $X_i' = X_i - \mu_i$ and $X' = X_1' + \ldots + X_n'$, then $\mathbf{P}[X \geq \mu + t] = \mathbf{P}[X' \geq t]$

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Let $X_1, \ldots, X_n$ be independent random variable with mean μ_i such that $a_i \leq X_i \leq b_i$. Let $X = X_1 + \ldots + X_n$, and let $\mu = \mathbf{E}[X] = \sum_{i=1}^n \mu_i$. Then for any $t > 0$

$$\mathbf{P}[X \geq \mu + t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right),$$

and

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Proof Outline (skipped):

- Let $X_i' = X_i - \mu_i$ and $X' = X_1' + \ldots + X_n'$, then $\mathbf{P}[X \geq \mu + t] = \mathbf{P}[X' \geq t]$
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This is not magic! you just need to optimise λ!

Method of Bounded Differences

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In all those cases (and more) we can easily prove concentration of $f(X_1, \ldots, X_n)$ around its mean by the so-called **Method of Bounded Differences**.

Method of Bounded Differences

A function f is called Lipschitz with parameters $\mathbf{c} = (c_1, \ldots, c_n)$ if for all $i = 1, 2, \ldots, n$,

$$|f(x_1, x_2, \ldots, x_{i-1}, \boldsymbol{x_i}, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, \boldsymbol{y_i}, x_{i+1}, \ldots, x_n)| \leq c_i,$$

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- The proof is omitted here (it requires the concept of martingales).

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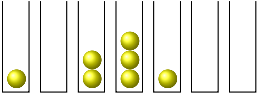
Application 3: Balls into Bins (again...)



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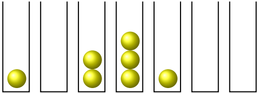

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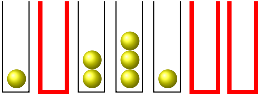

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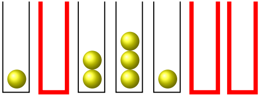

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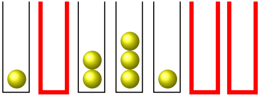

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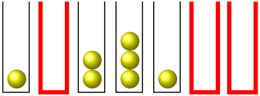

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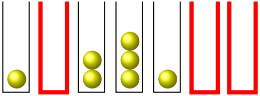

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This is a decent bound, but for some values of m it is far from tight and stronger bounds are possible through a refined analysis.

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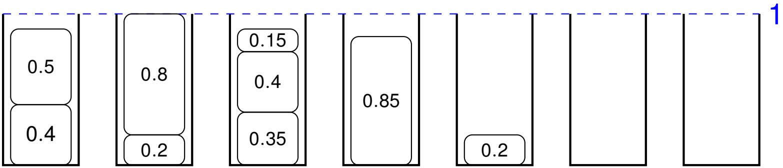

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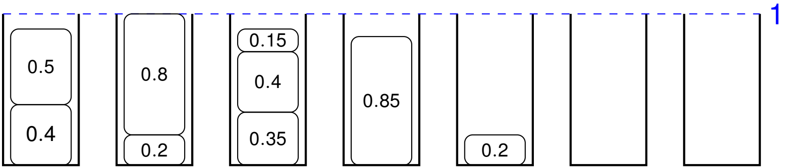

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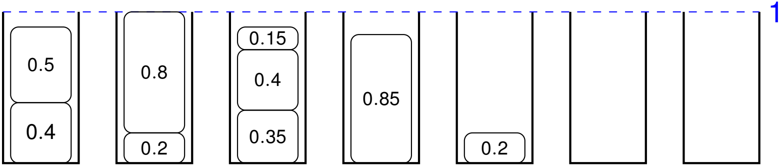

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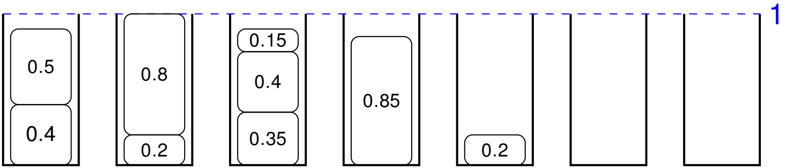

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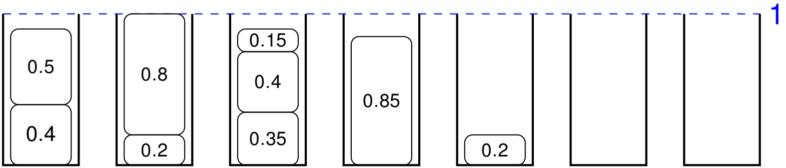

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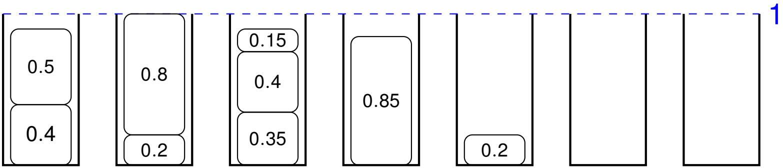

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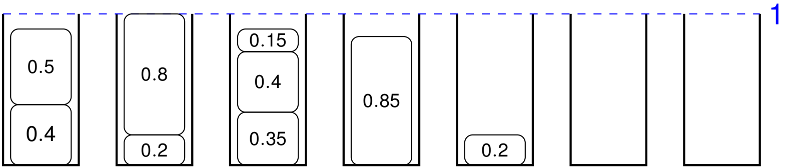

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This is a typical example where proving concentration is much easier than calculating (or estimating) the expectation!

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Proof of 2:

$$M_{X+Y}(t) = \mathbf{E}\left[e^{t(X+Y)}\right] = \mathbf{E}\left[e^{tX} \cdot e^{tY}\right] \stackrel{(!)}{=} \mathbf{E}\left[e^{tX}\right] \cdot \mathbf{E}\left[e^{tY}\right] = M_X(t)M_Y(t) \qquad \square$$