Randomised Algorithms

Lecture 15: Bandit Algorithms

Thomas Sauerwald (tms41@cam.ac.uk)

Lent 2022

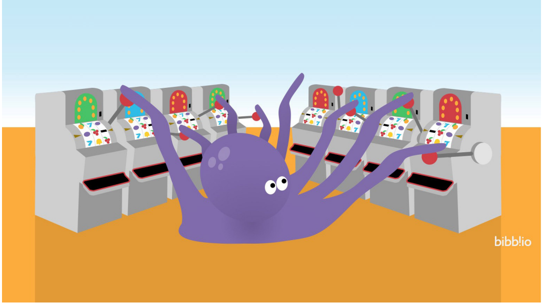
Outline

Introduction

Stochastic Bandits

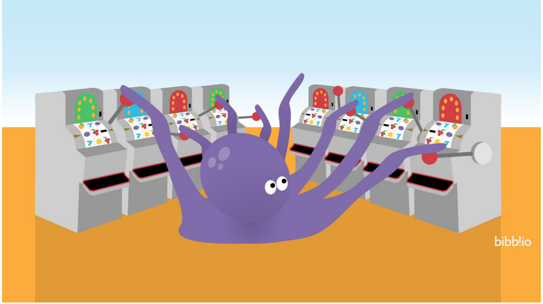
Outlook: Adversarial Bandits (non-examinable)

Multi-Armed Bandits



Source: Bibblio

Multi-Armed Bandits



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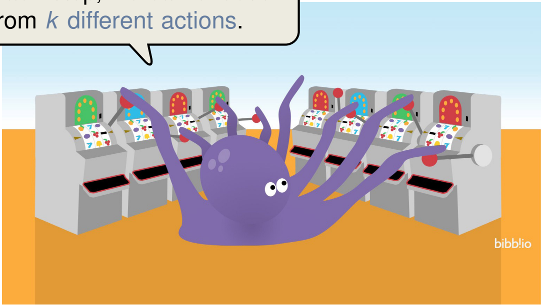
Multi-Armed Bandits: make a sequence of decisions under uncertainty.

Multi-Armed Bandits

At each step, we can choose from *k* different actions.



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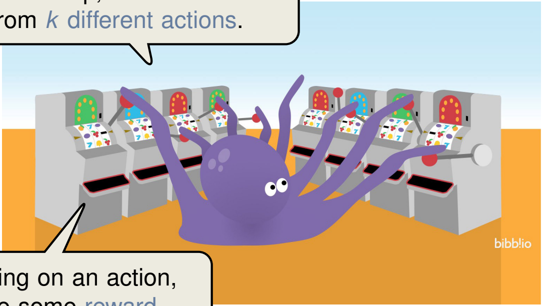

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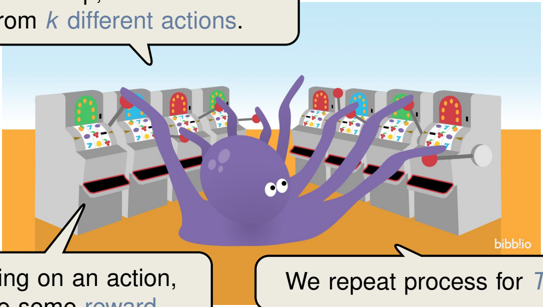

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We repeat process for *T* steps.

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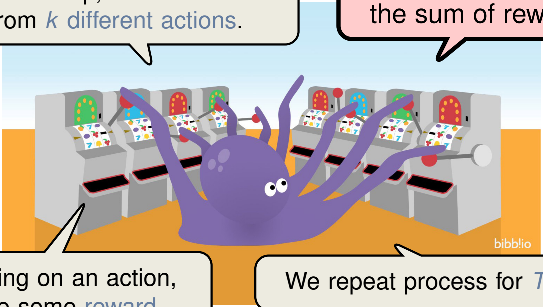
How can we maximise the sum of rewards?



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Bandit Model versus Expert Model

In the Online Learning using Expert setting:

- We have n experts and at each round each expert makes a prediction, which may be correct or wrong
- Our goal is to make a prediction at each round and perform (almost) as good as the best expert.

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There is a rich interplay between the two models (see EXP3 algorithm later)!

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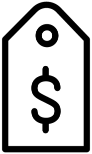

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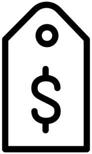

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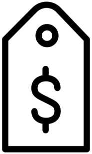

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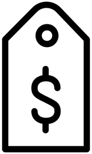
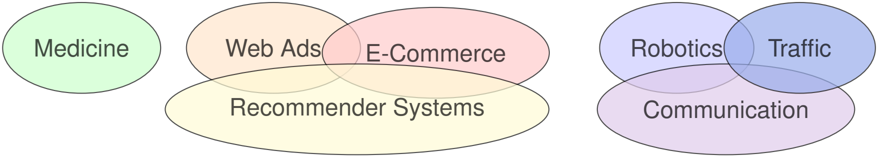


Applications of Multi-Armed Bandits (2/2)

Application domain	Action	Reward
medical trials	which drug to prescribe	health outcome.
web design	*e.g.,* font color or page layout	#clicks.
content optimization	which items/articles to emphasize	#clicks.
web search	search results for a given query	1 if the user is satisfied.
advertisement	which ad to display	revenue from ads.
recommender systems	*e.g.,* which movie to watch	1 if follows recommendation.
sales optimization	which products to offer at which prices	revenue.
procurement	which items to buy at which prices	#items procured
auction/market design	*e.g.,* which reserve price to use	revenue
crowdsourcing	which tasks to give to which workers, and at which prices	1 if task completed at sufficient quality.
datacenter design	*e.g.,* which server to route the job to	job completion time.
Internet	*e.g.,* which TCP settings to use?	connection quality.
radio networks	which radio frequency to use?	1 if successful transmission.
robot control	a "strategy" for a given task	job completion time.

Source: Survey by Slivkins



Types of Multi-Armed Bandits Environments

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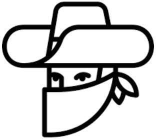

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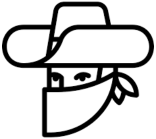

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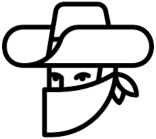

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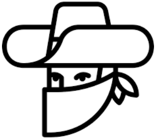

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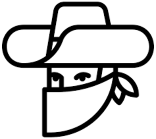


Applications of Contextual Bandits

How The New York Times is Experimenting with Recommendation Algorithms

NYT Open

Algorithmic curation at The Times is used in designated parts of our website and apps.

Anna Coenen Follow
Oct 17, 2019 · 6 min read

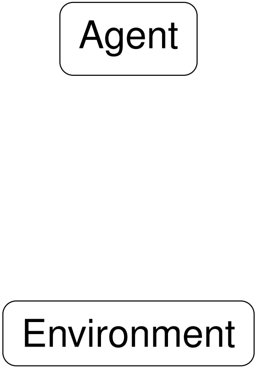
A contextual recommendation approach

One recommendation approach we have taken uses a class of algorithms called contextual multi-armed bandits. Contextual bandits learn over time how people engage with particular articles. They then recommend articles that they predict will garner higher engagement from readers. The *contextual* part means that these bandits can use additional information to get a better estimate of how engaging an article might be to a particular reader. For example, they can take into account a reader's geographical region (like country or state) or reading history to decide if a particular article would be relevant to that reader.

["recommended": "article B"; "reader state": "Texas", "clicked": "yes"]
["recommended": "article A"; "reader state": "New York", "clicked": "yes"]
["recommended": "article B"; "reader state": "New York", "clicked": "no"]
["recommended": "article B"; "reader state": "California", "clicked": "no"]
["recommended": "article A"; "reader state": "New York", "clicked": "no"]

Once the bandit has been trained on the initial data, it might suggest Article A, Article B or a new article, C, for a new reader from New York. The bandit would be most likely to recommend Article A because the article had the highest click-through rate with New York readers in the past. With some smaller probability, it might also try showing Article C, because it doesn't yet know how engaging it is and needs to generate some data to learn about it.

Online Algorithm/Reinforcement Learning Framework



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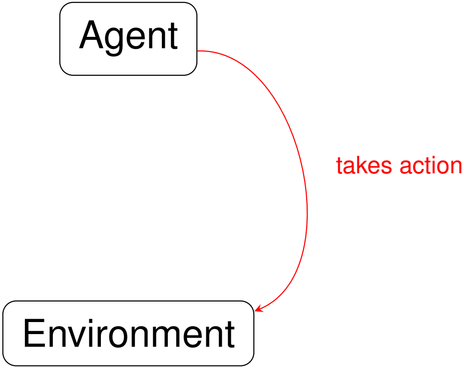

Iteration: 1

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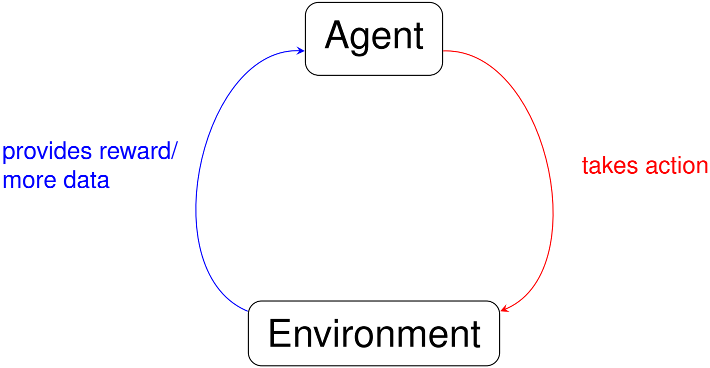

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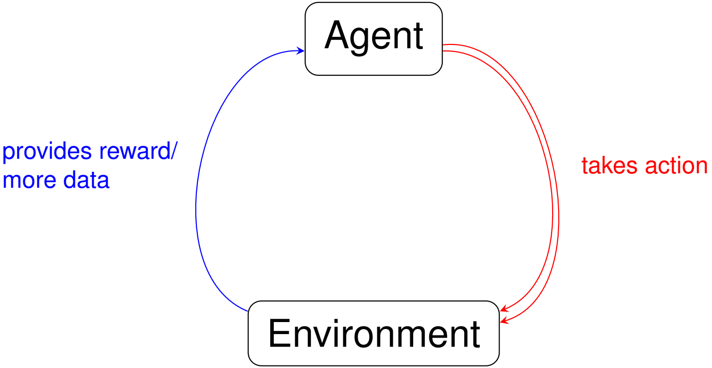

Iteration: 2

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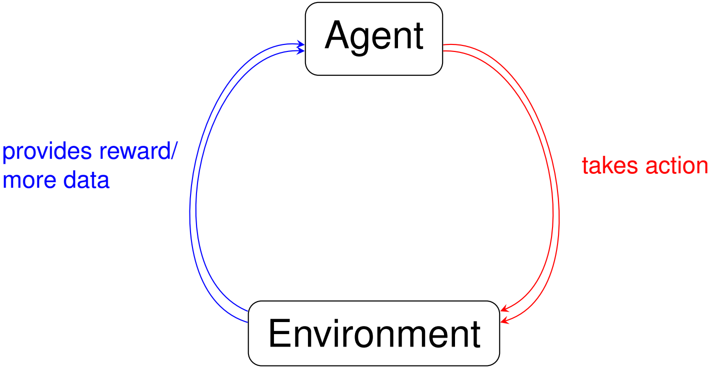

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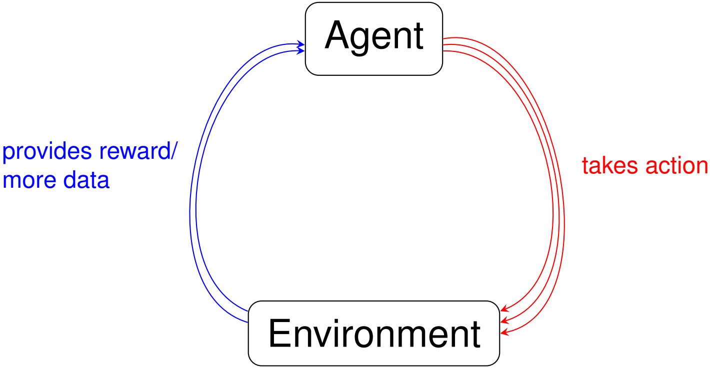

Iteration: 3

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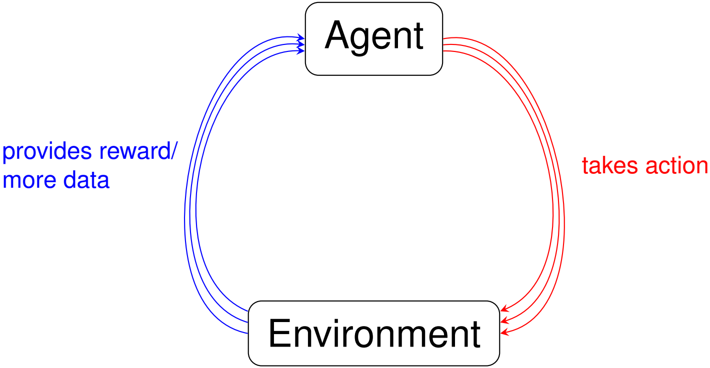

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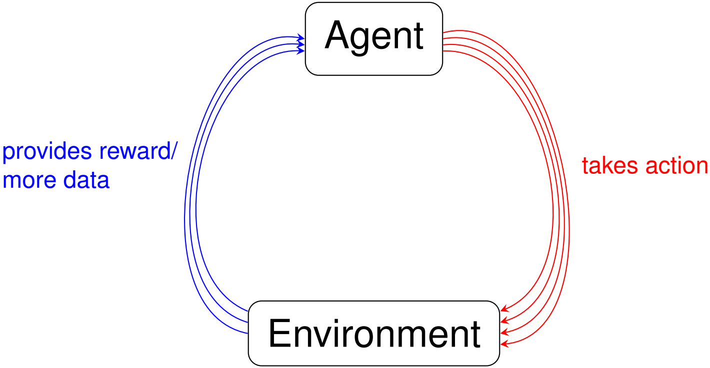

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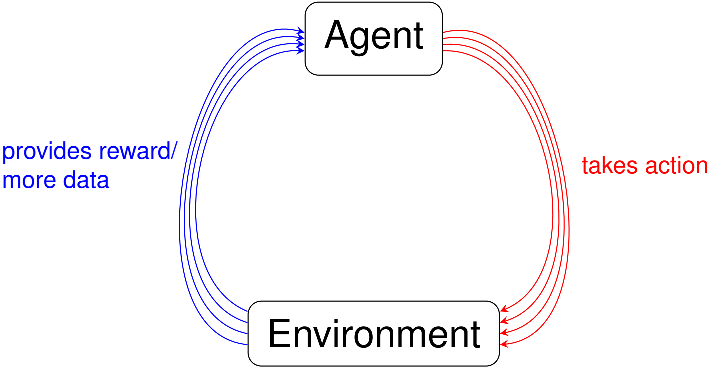

Iteration: 4

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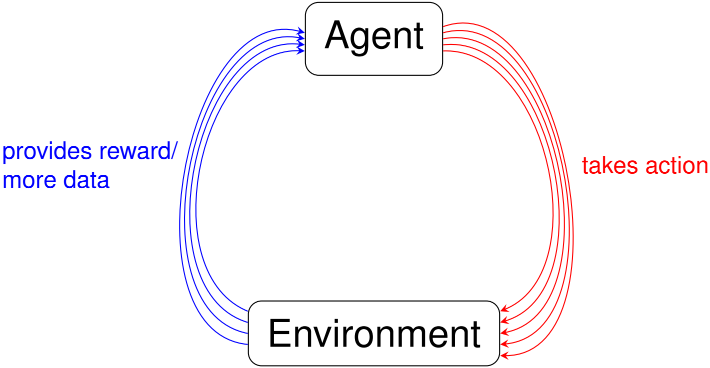

Iteration: 5

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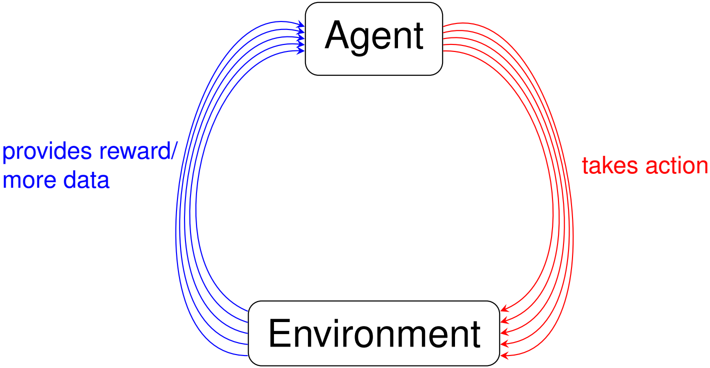

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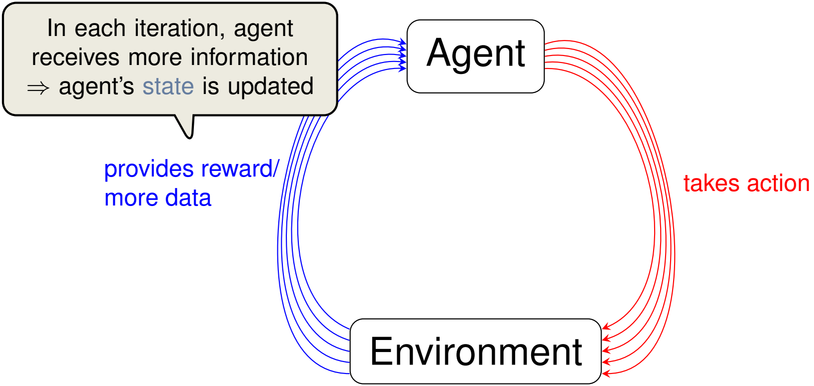

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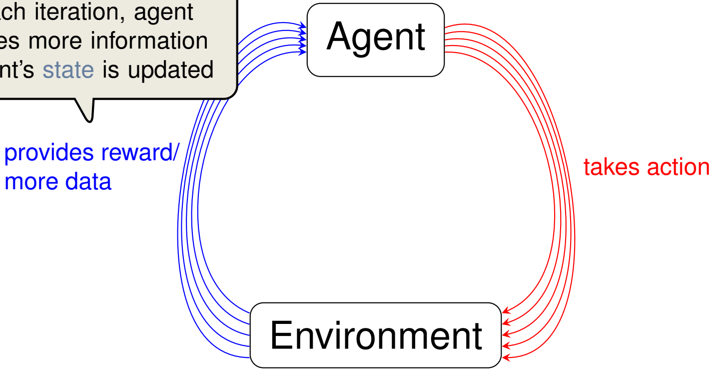
Online Algorithm/Reinforcement Learning Framework

Exploration vs. Exploitation

In each iteration, agent receives more information
⇒ agent's state is updated



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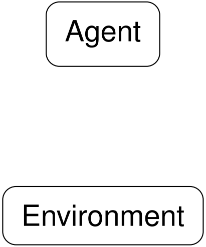

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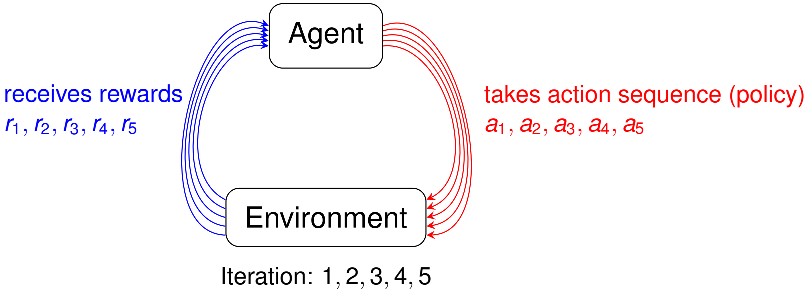

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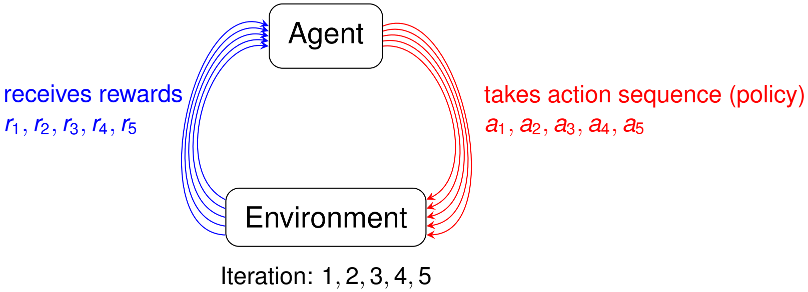

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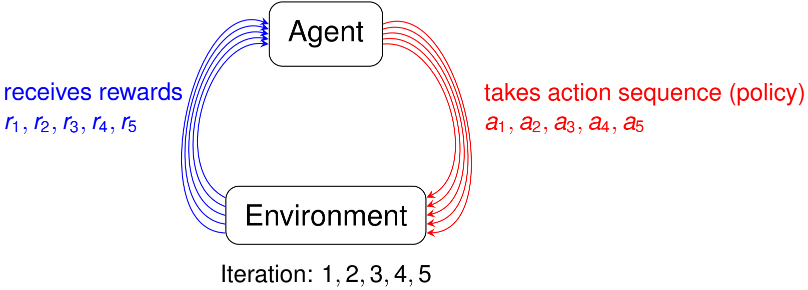

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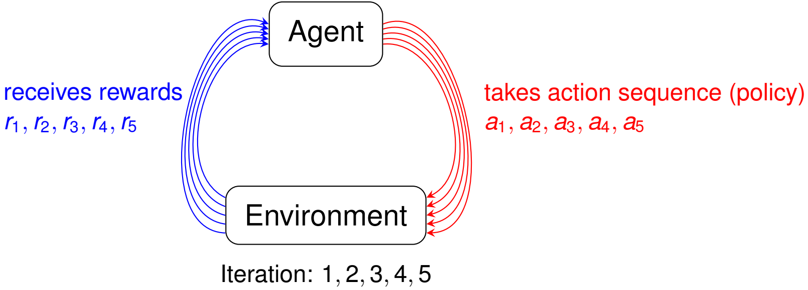

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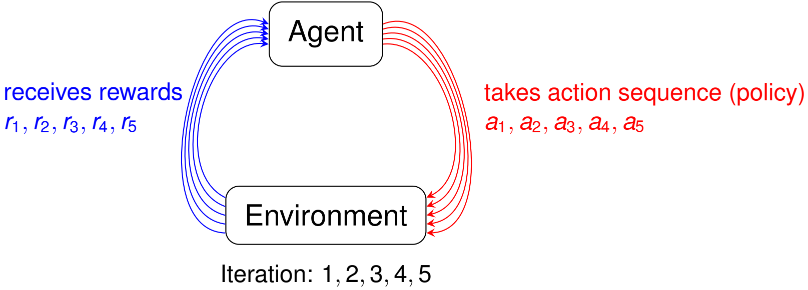

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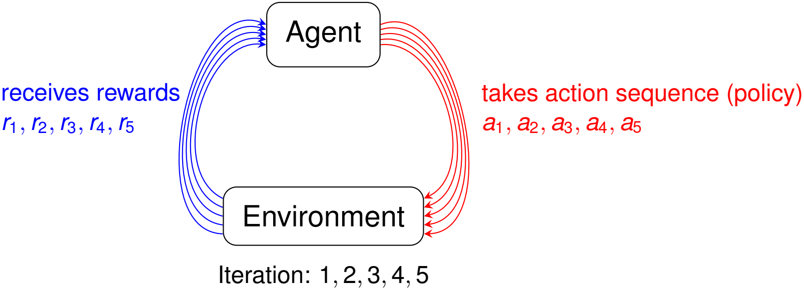

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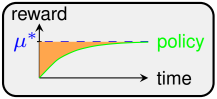

Comparing against the mean-optimal strategy ("best-arm benchmark")

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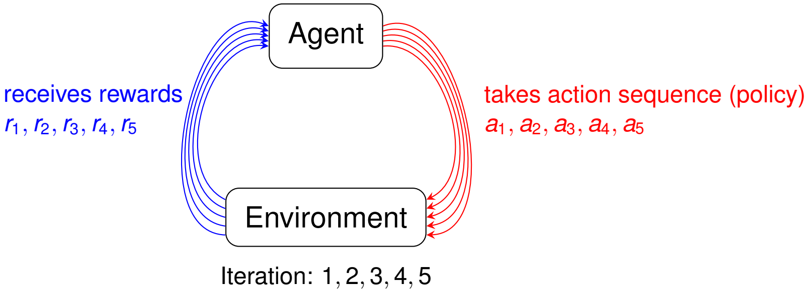

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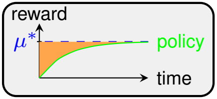


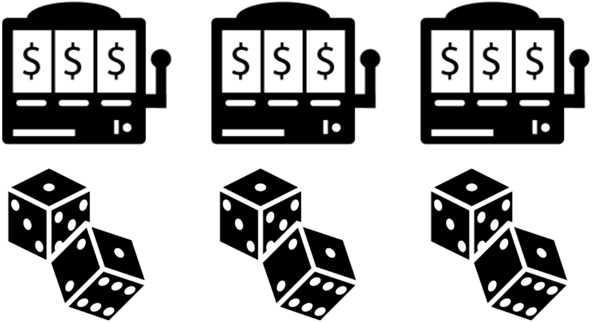
$$R_T(\pi) = T \cdot \mu^* - \sum_{t=1}^{T} \mu(a_t).$$

Comparing against the mean-optimal strategy ("best-arm benchmark")

This sum depends on the policy π and horizon T, but it is deterministic.

Stochastic (Bernoulli) Bandits

- Consider the time-horizon $1, 2, \ldots, T$
- We have k different actions (arms) at each step
- Every reward is a binary random variable with unknown probability



Stochastic (Bernoulli) Bandits

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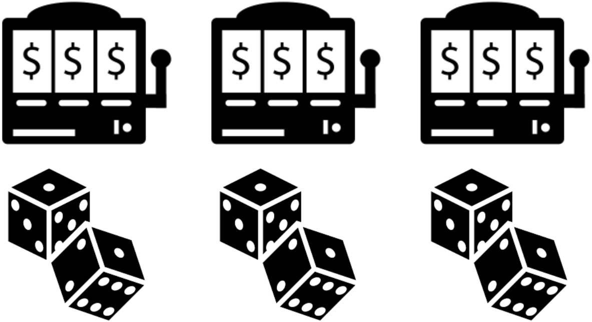
This is also known as Bernoulli Bandits



Regret in Bernoulli Bandits: Example

Let $k = 3$ 

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t	Available Actions	Reward	Total (Realised) Reward
1	1, 2, 3		

Regret in Bernoulli Bandits: Example

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t	Available Actions	Reward	Total (Realised) Reward
1	1, 2, 3	0	0

Regret in Bernoulli Bandits: Example

Let $k = 3$

t	Available Actions	Reward	Total (Realised) Reward
1	1, 2, 3	0	0
2	1, 2, 3		

Regret in Bernoulli Bandits: Example

Let $k = 3$

t	Available Actions	Reward	Total (Realised) Reward
1	1, 2, 3	0	0
2	1, 2, 3	1	1

Regret in Bernoulli Bandits: Example

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Regret in Bernoulli Bandits: Example

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t	Available Actions	Reward	Total (Realised) Reward
1	1, 2, 3	0	0
2	1, 2, 3	1	1
3	1, 2, 3	1	2

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t	Available Actions	Reward	Total (Realised) Reward
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2	1, 2, 3	1	1
3	1, 2, 3	1	2
4	1, 2, 3		

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4	1, 2, 3	0	2

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4	1, 2, 3	0	2
5	1, 2, 3	1	3

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5	1, 2, 3	1	3
6	1, 2, 3	0	3
7	1, 2, 3	1	4

Regret in Bernoulli Bandits: Example

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t	Available Actions	Reward	Total (Realised) Reward
1	1, 2, 3	0	0
2	1, 2, 3	1	1
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Regret in Bernoulli Bandits: Example

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t	Available Actions	Reward	Total (Realised) Reward
1	**1**, 2, 3	0	0
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3	1, **2**, 3	1	2
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6	1, 2, **3**	0	3
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5	1, 2, 3	1	3
6	1, 2, 3	0	3
7	1, 2, 3	1	4
8	1, 2, 3	0	4
9	1, 2, 3	1	5

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1	1, 2, 3	0	0
2	1, 2, 3	1	1
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4	1, 2, 3	0	2
5	1, 2, 3	1	3
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Exercise: Assume $\mu(1) = 0.4$, $\mu(2) = 0.5$, $\mu(3) = 0.7$. What is the regret?

1. Compute maximal mean reward $T \cdot \mu^*$
2. Compute mean reward of used policy $(1, 2, 2, 2, 3, 3, 2, 2, 3, 3)$

Regret in Bernoulli Bandits: Example

Let $k = 3$

t	Available Actions	Reward	Total (Realised) Reward
1	**1**, 2, 3	0	0
2	1, **2**, 3	1	1
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Regret in Bernoulli Bandits: Example

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1	$1, 2, 3$	0	0
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8	$1, 2, 3$	0	4
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1. Maximal mean reward is $T \cdot \mu^* = 10 \cdot 0.7 = 7$

Regret in Bernoulli Bandits: Example

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1	1, 2, 3	0	0
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2. Mean reward of our policy is $1 \cdot 0.4 + 5 \cdot 0.5 + 4 \cdot 0.7 = 5.7$

Regret in Bernoulli Bandits: Example

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1	1, 2, 3	0	0
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1. Maximal mean reward is $T \cdot \mu^* = 10 \cdot 0.7 = 7$
2. Mean reward of our policy is $1 \cdot 0.4 + 5 \cdot 0.5 + 4 \cdot 0.7 = 5.7$

$\Rightarrow$ Cumulative Regret is $7 - 5.7 = 1.3$

Question: Why does regret involve mean rewards and not realised rewards?

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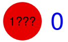

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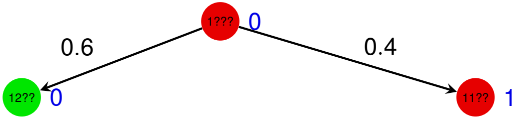

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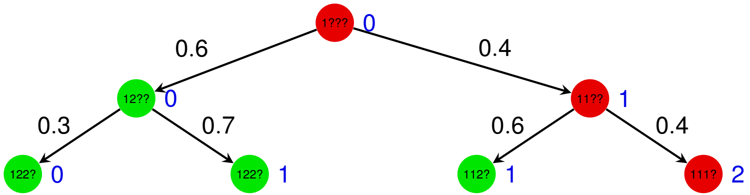

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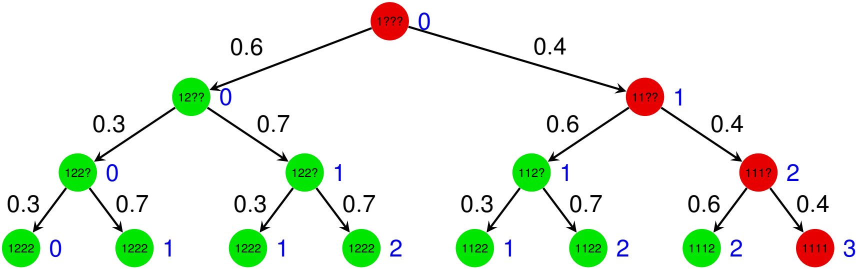

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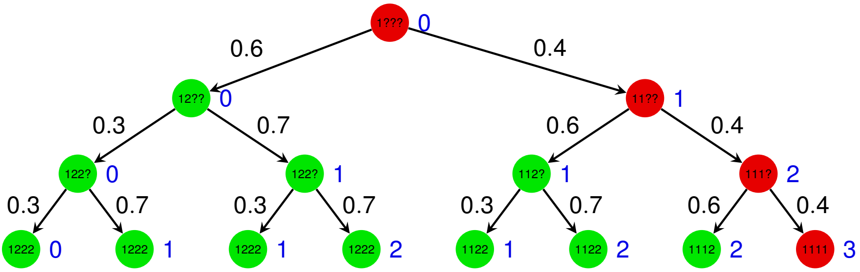
- Expected realised reward is the (weighted) average over the rewards of the 8 leafs



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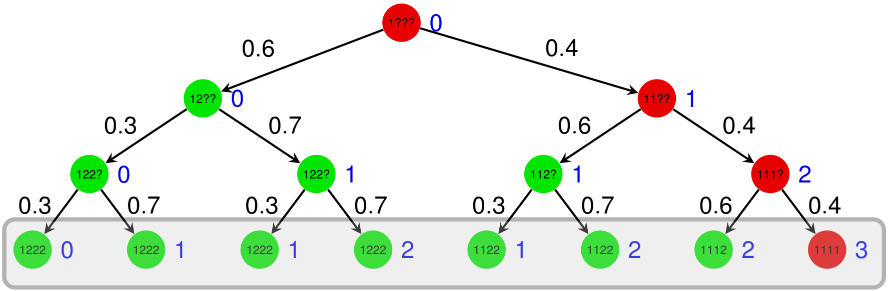

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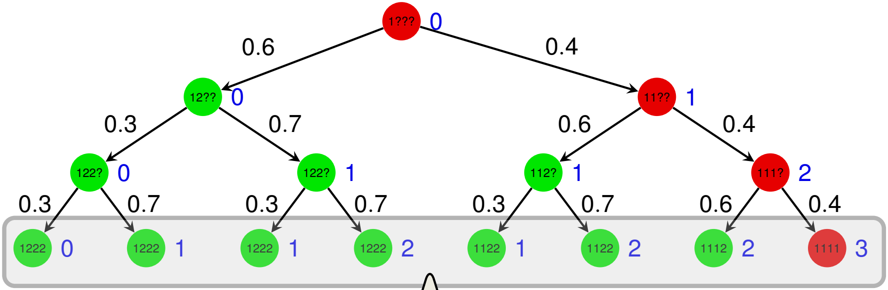
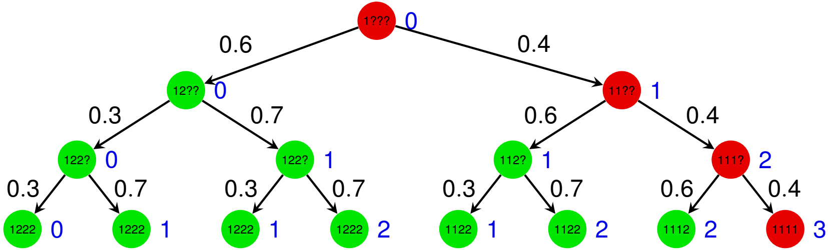
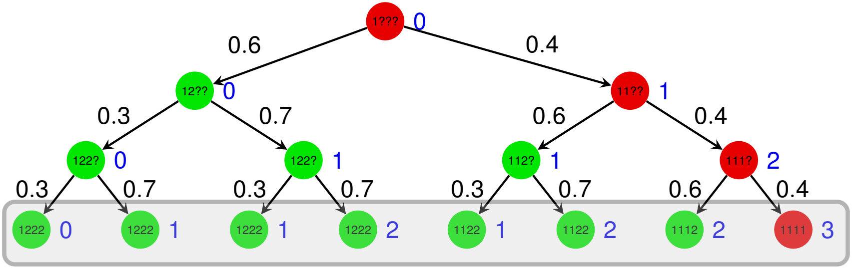


Expected Reward is: $0.6 \cdot 0.3 \cdot 0.3 \cdot 0 + 0.6 \cdot 0.3 \cdot 0.7 \cdot 1 + \ldots$


1??? 0
0.6
0.4
12?? 0
11?? 1
0.3
0.7
0.6
0.4
122? 0
122? 1
112? 1
111? 2
0.3
0.7
0.3
0.7
0.3
0.7
0.6
0.4
1222 0
1222 1
1222 1
1222 2
1122 1
1122 2
1112 2
1111 3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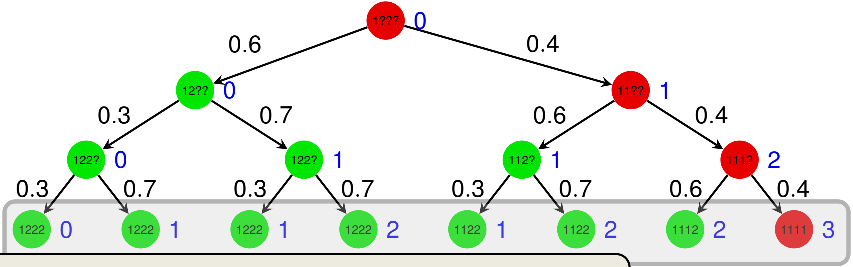
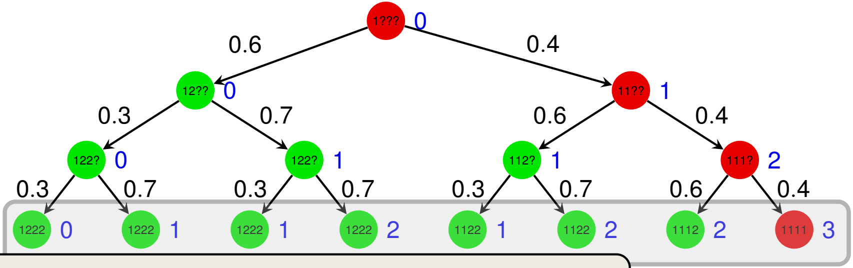


- Let us change the reward calculation to mean reward!



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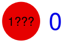

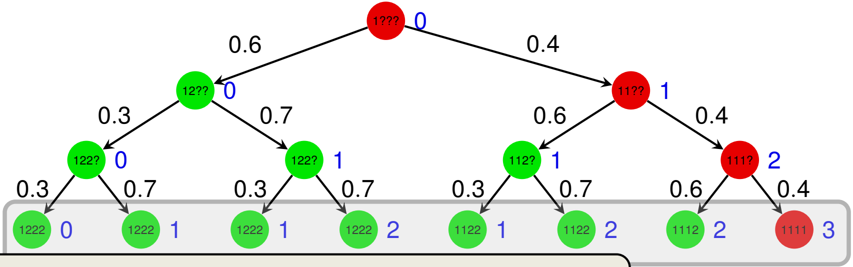

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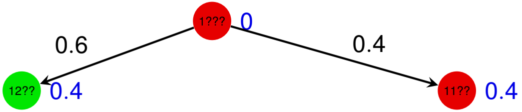

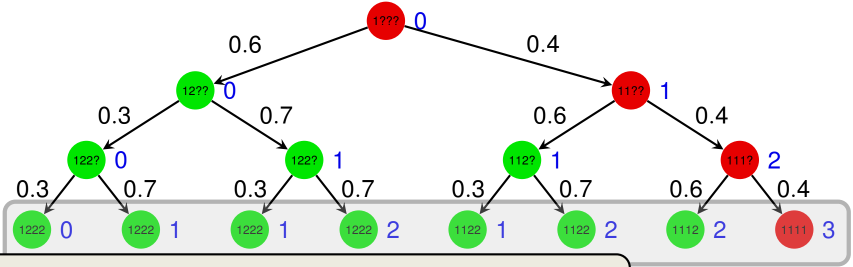

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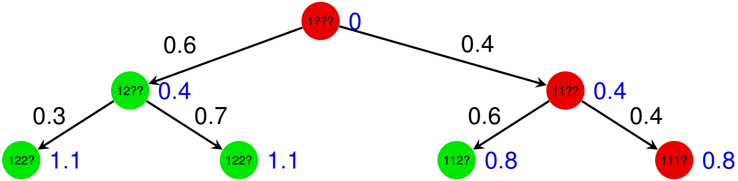

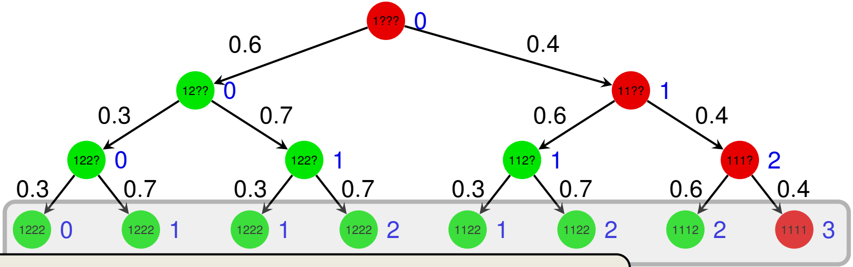

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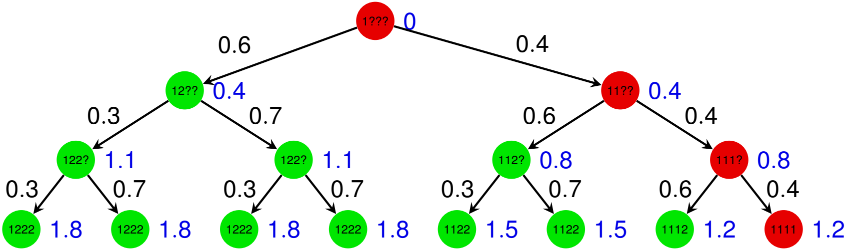
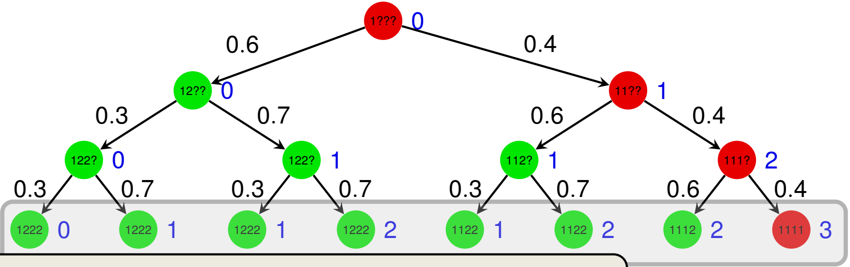
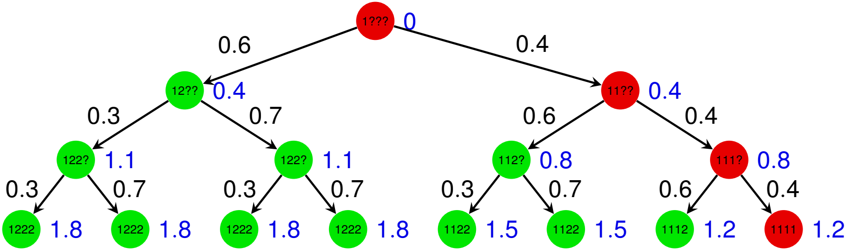
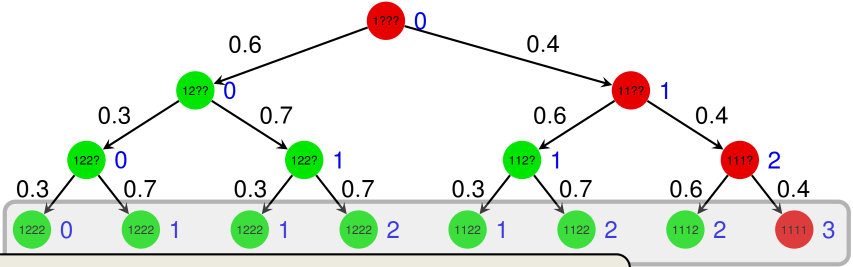




- Let us change the reward calculation to mean reward!
- The expected reward is now easier to compute:





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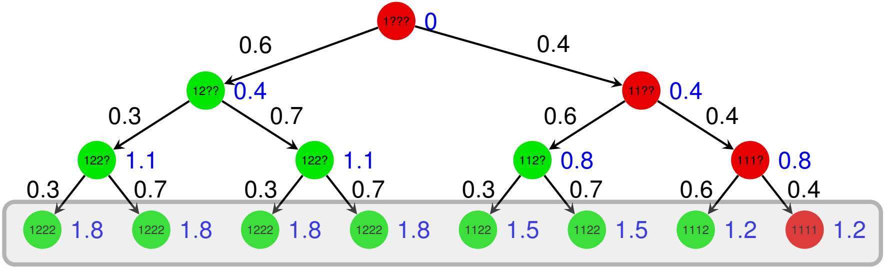
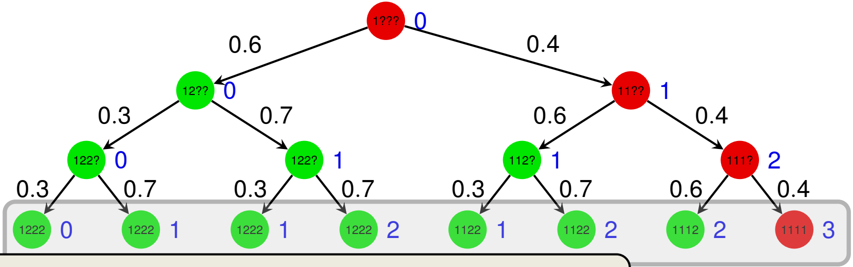
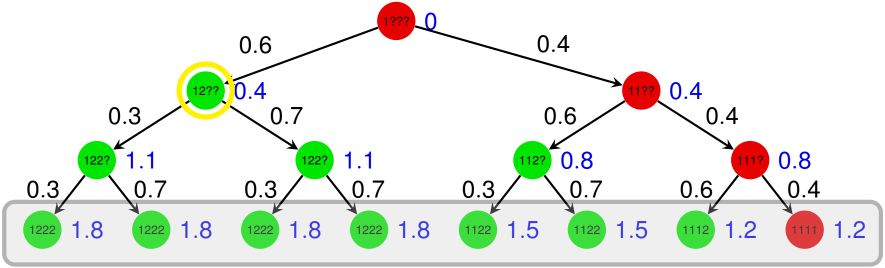
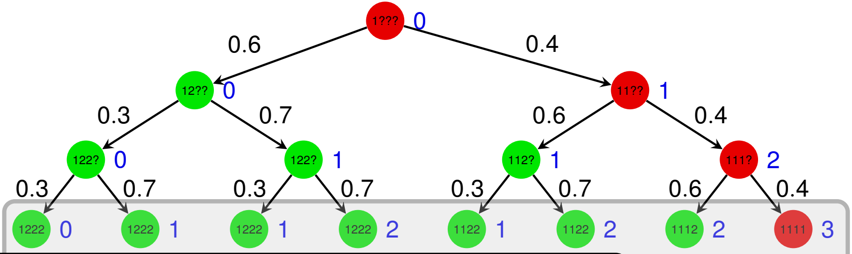


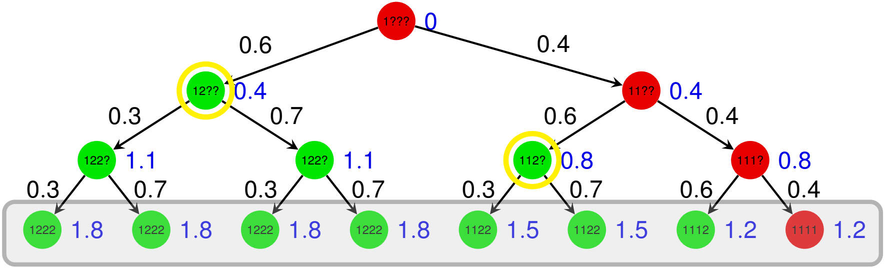


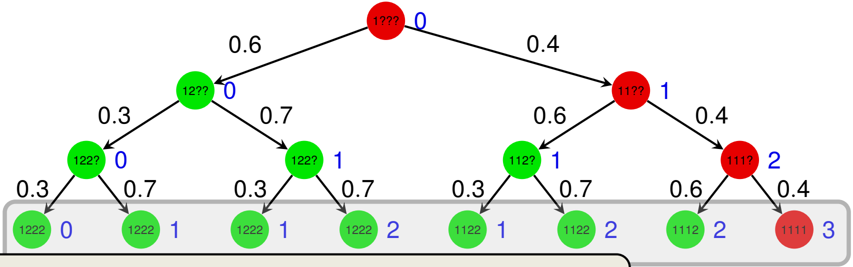
- Let us change the reward calculation to mean reward!
- The expected reward is now easier to compute: $0.6 \cdot 1.8$



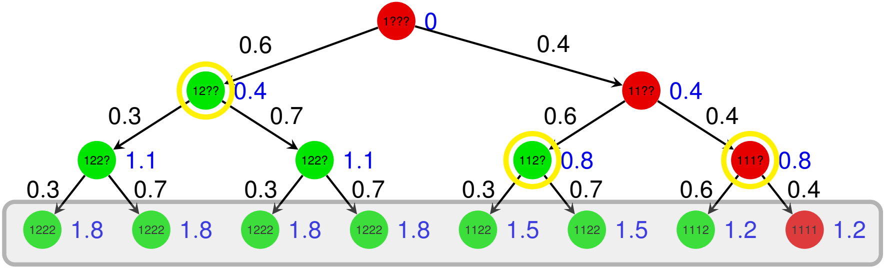


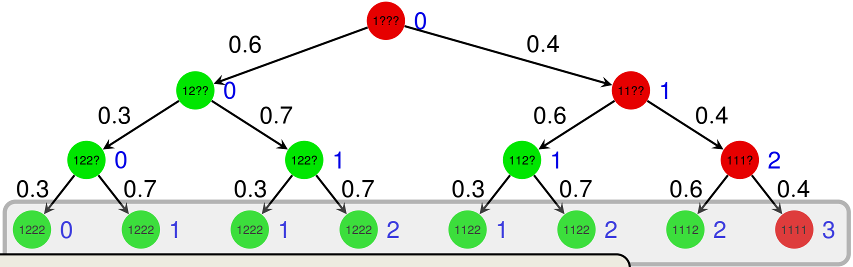
- Let us change the reward calculation to mean reward!
- The expected reward is now easier to compute:
  $0.6 \cdot 1.8 + 0.4 \cdot 0.6 \cdot 1.5$





- Let us change the reward calculation to mean reward!
- The expected reward is now easier to compute:
  $0.6 \cdot 1.8 + 0.4 \cdot 0.6 \cdot 1.5 + 0.4 \cdot 0.4 \cdot 1.2$





- Let us change the reward calculation to mean reward!
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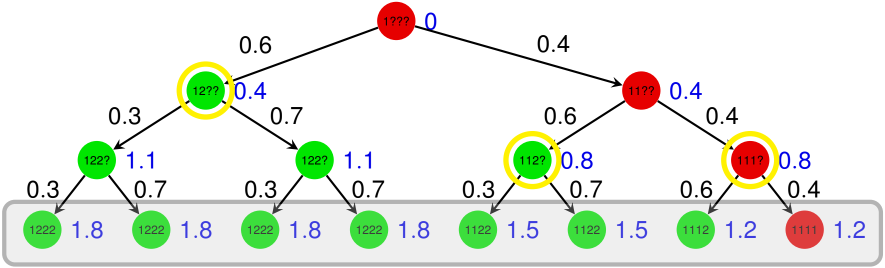

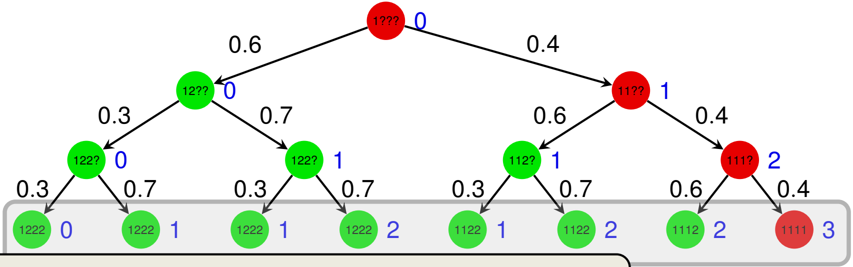

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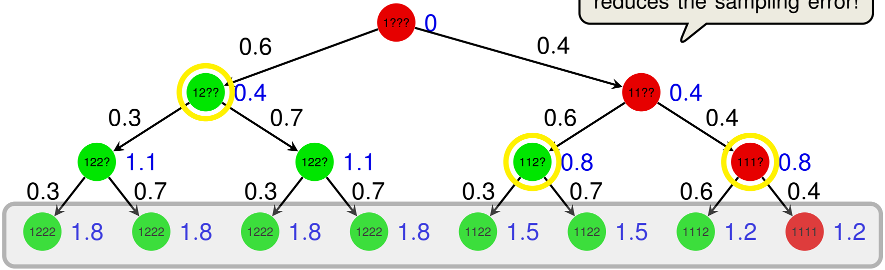
Using mean reward reduces the sampling error!



A Simple Heuristic

Algorithm 1: Greedy

- Idea: Choose the arm with the highest average realised reward so far

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1. That is, for every action a and step t, we compute

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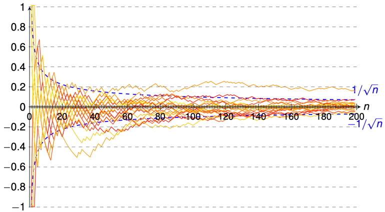
This is a general method called action-value method: guide decisions by estimating values of actions



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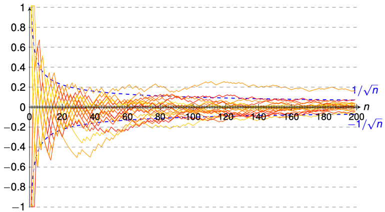

Exercise: Do you think this is a good strategy?

Regret in Bernoulli Bandits: Greedy on Earlier Example

Let $k = 3$ and $\mu(1) = 0.4$, $\mu(2) = 0.5$, $\mu(3) = 0.7$.

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1. Greedy will in the long run achieve reward of $T \cdot 0.5$

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1. Greedy will in the long run achieve reward of $T \cdot 0.5$
2. Greedy will never try action 3, which is better! **Not enough exploration!**

Improving Greedy

Algorithm 2: ϵ-Greedy

- Idea: With probability $\epsilon \in (0, 1)$ pick an action uniformly at random, otherwise perform Greedy

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Hence the algorithm will eventually "learn" optimal policy and the regret is small.

How should we choose ϵ in order to minimise the regret?

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Regret in Bernoulli Bandits: Example of ϵ-Greedy

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2	1, 2, 3	1	1	0.9
3	1, 2, 3	0	1	1.3
4	1, 2, 3	0	1	1.8
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1. ϵ-Greedy may take a lot of sub-optimal actions at the beginning
2. However, it explores all actions often enough!

Experimental Results: Greedy and ϵ-Greedy (1/2)

To roughly assess the relative effectiveness of the greedy and ε-greedy action-value methods, we compared them numerically on a suite of test problems. This was a set of 2000 randomly generated k-armed bandit problems with $k = 10$. For each bandit problem, such as the one shown in Figure 2.1, the action values, $q_*(a)$, $a = 1, \ldots, 10$,

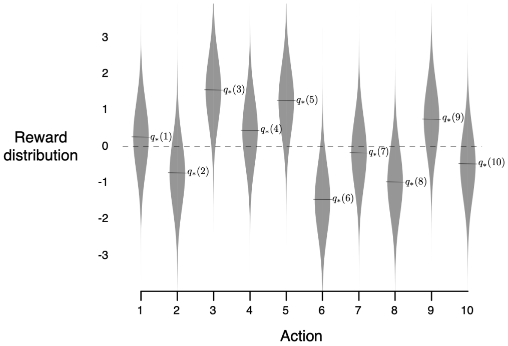
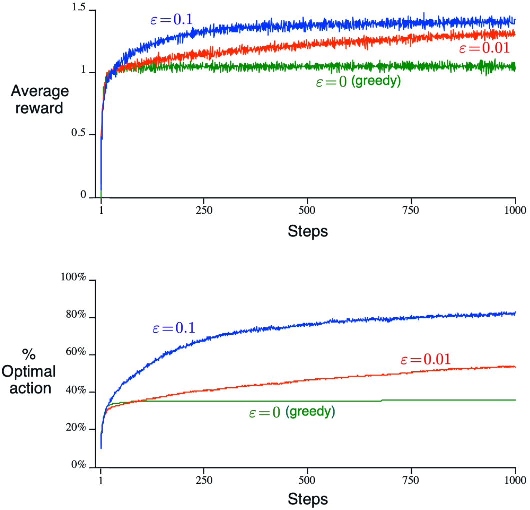


Figure 2.1: An example bandit problem from the 10-armed testbed. The true value $q_*(a)$ of each of the ten actions was selected according to a normal distribution with mean zero and unit variance, and then the actual rewards were selected according to a mean $q_*(a)$, unit-variance normal distribution, as suggested by these gray distributions.

Source: Sutton and Barto

Experimental Results: Greedy and ϵ-Greedy (2/2)



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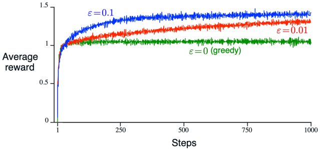
Exercise: What happens if $\epsilon_t = 1/t^2$?

Summary so far...



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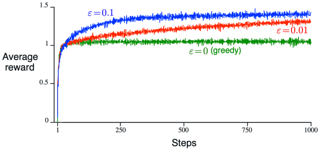

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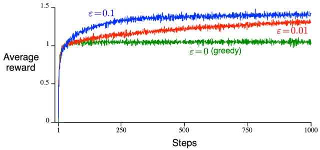

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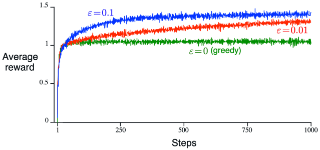

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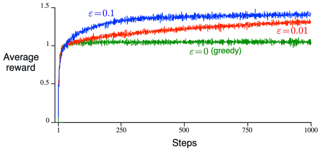

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Ideas for Improvements:

- In an exploration step, sample non-uniformly
- Blend exploration and exploitation by maintaining for each arm an **upper confidence bound** for the mean reward

Towards the UCB Algorithm

Question: How close are $Q_t(a)$ (the empirical estimate) and $\mu(a)$?

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- For each action, construct an optimistic guess for the expected reward
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$\Rightarrow$ generally prefer arms with high empirical reward and/or high uncertainty

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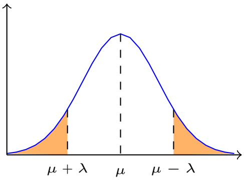

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Let $X_1, \ldots, X_n$ be n independent Bernoulli random variables. Let $X := \sum_{i=1}^n X_i$ and $\mu = \mathbf{E}[X]$. Then, for any $\lambda \geq 0$,

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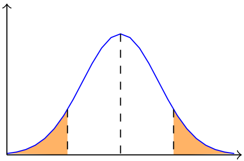

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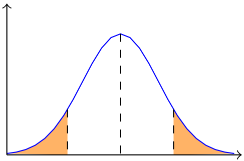

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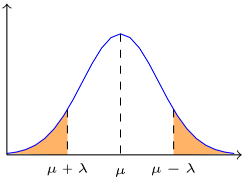

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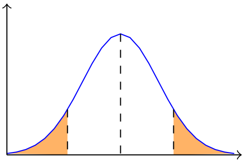
Make sure Confidence Bound always holds: $\delta = 2t^{-2}$.



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Let $X_1, \ldots, X_n$ be n independent Bernoulli random variables. Let $X := \sum_{i=1}^{n} X_i$ and $\mu = \mathbf{E}[\,X\,]$. Then, for any $\lambda \geq 0$,

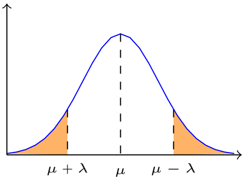
$$\mathbf{P}\left[\,|X - \mu| \geq \lambda\,\right] \leq 2 \cdot \exp\left(-\frac{2\lambda^2}{n}\right).$$

In our application, $n = n_t(a) := \sum_{i=1}^{t} \mathbf{1}_{a_i = a}$, $X = Q_t(a) \cdot n$.

Chernoff Bounds

Question: How close are $Q_t(a)$ (the empirical estimate) and $\mu(a)$?

Want Confidence-Bound like $\mathbf{P}\left[\,|Q_t(a) - \mu(a)| \leq \Delta_t(a)\,\right] \geq 1 - \delta$



Make sure Confidence Bound always holds: $\delta = 2t^{-2}$.

$\lambda(a) = \sqrt{n_t(a) \cdot \log(t)}$

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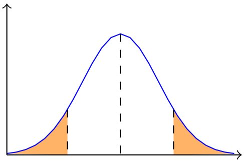

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The UCB Algorithm

Algorithm 3: UCB Algorithm

Initialisation: Let $n_1(a) = 0$ and $Q_1(a) = 0$ for all actions a
Execute: For $t = 1, 2, \ldots, T$:

- Take a that maximises $\widetilde{\mu}(a) = Q_t(a) + \sqrt{\frac{\log(t)}{n_t(a)}}$ and receive reward r_t
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$$n_{t+1}(a) \leftarrow n_t(a) + 1$$

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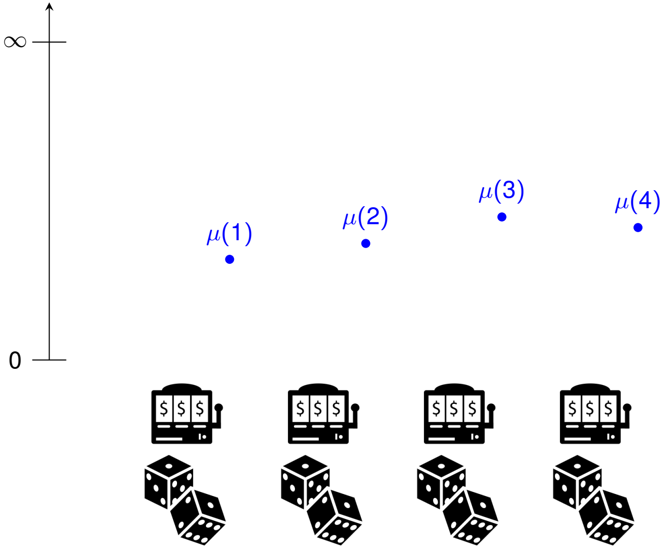
UCB-Algo takes sub-optimal actions only at a logarithmic rate!

Example 1: Illustration of UCB (simplified)



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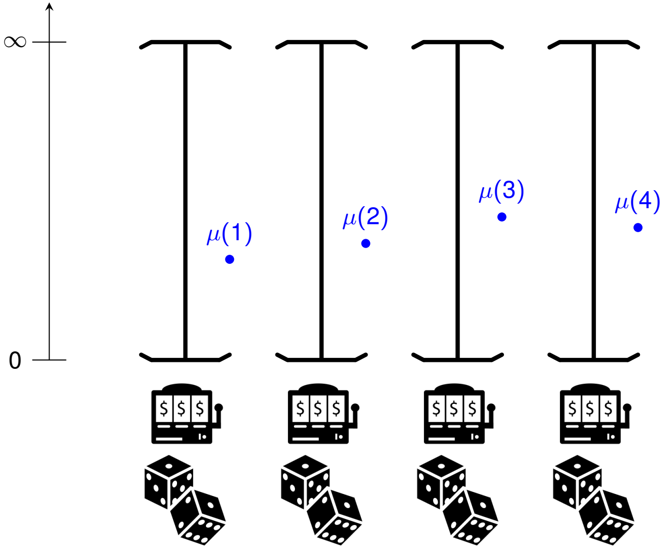

Step: 1

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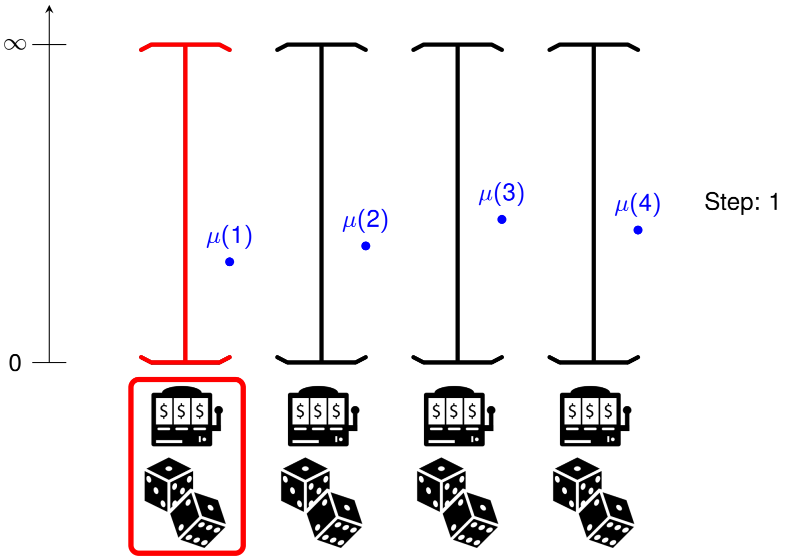

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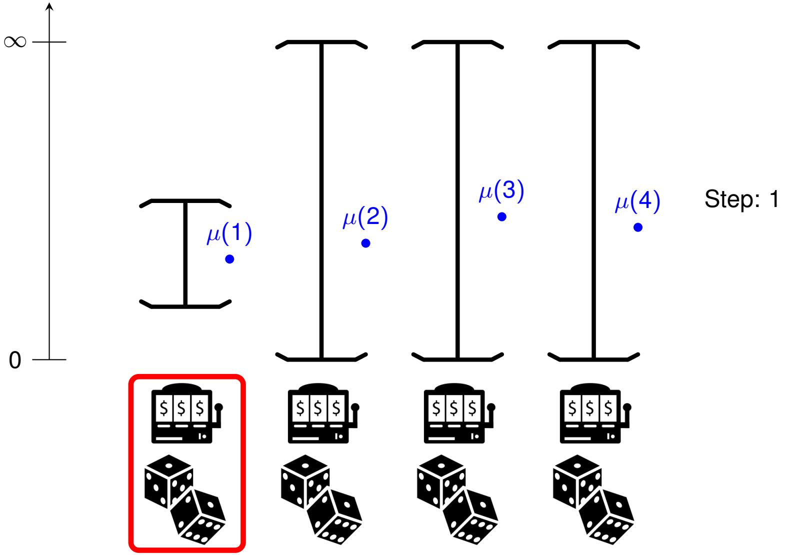

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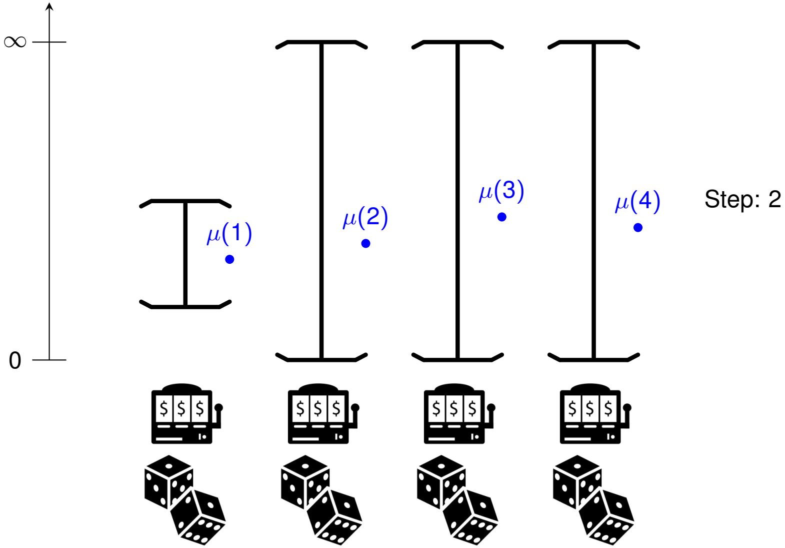

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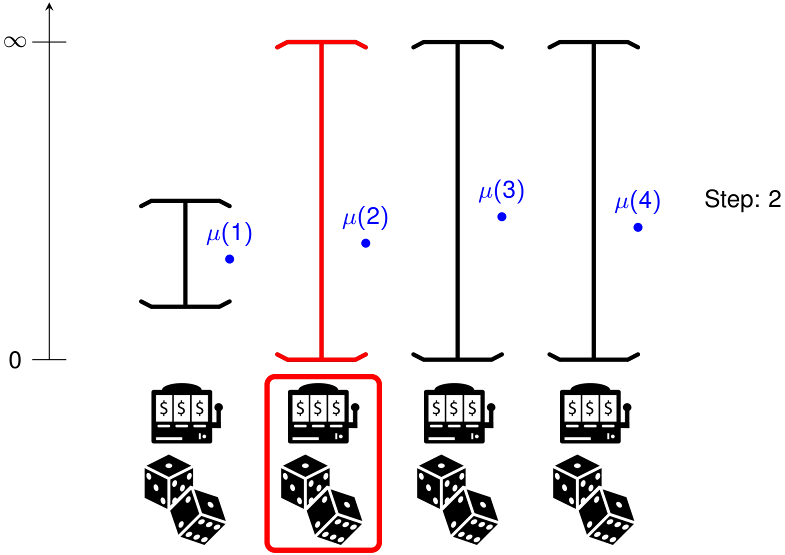

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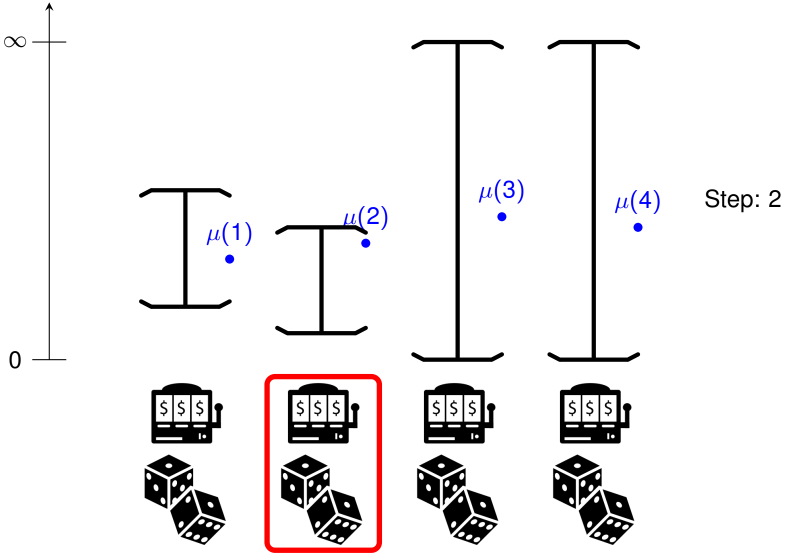

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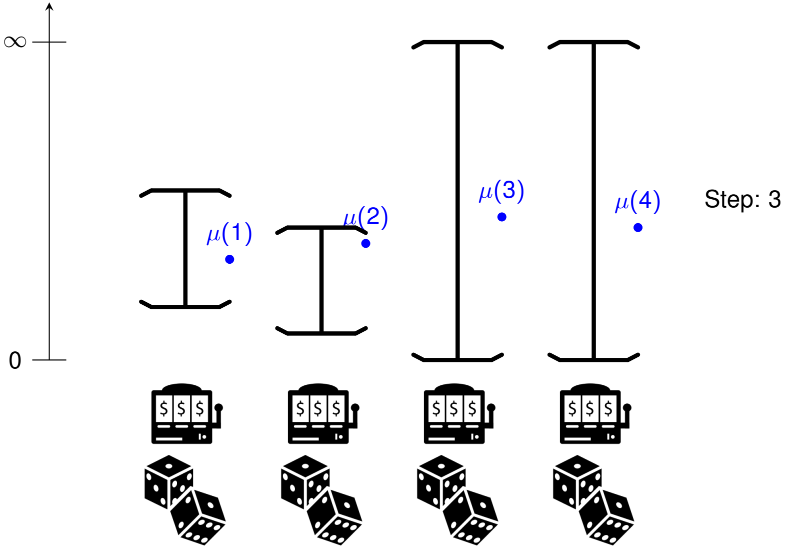

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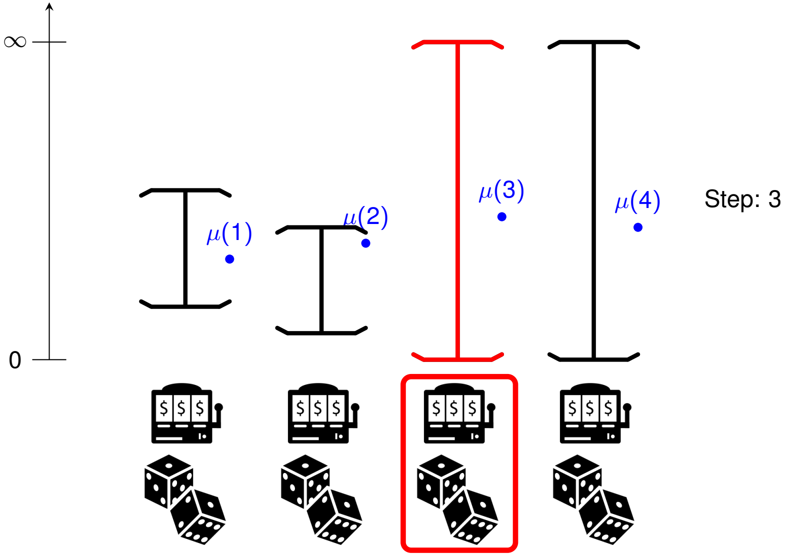

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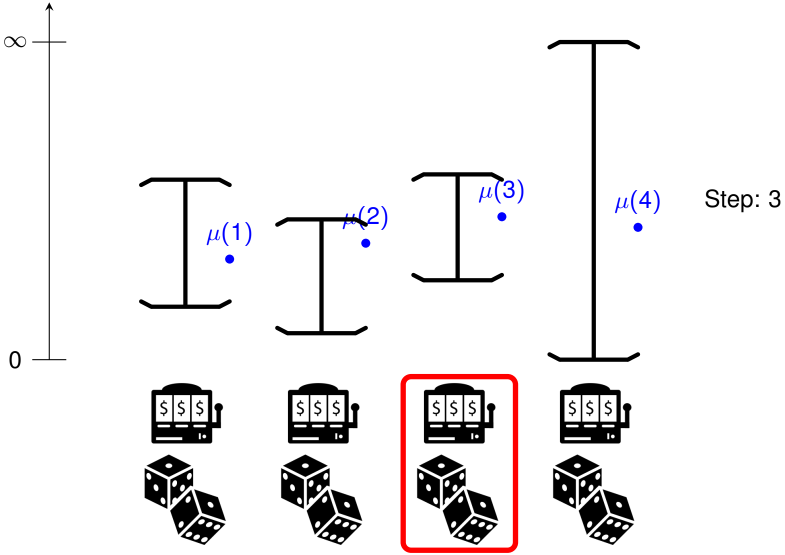

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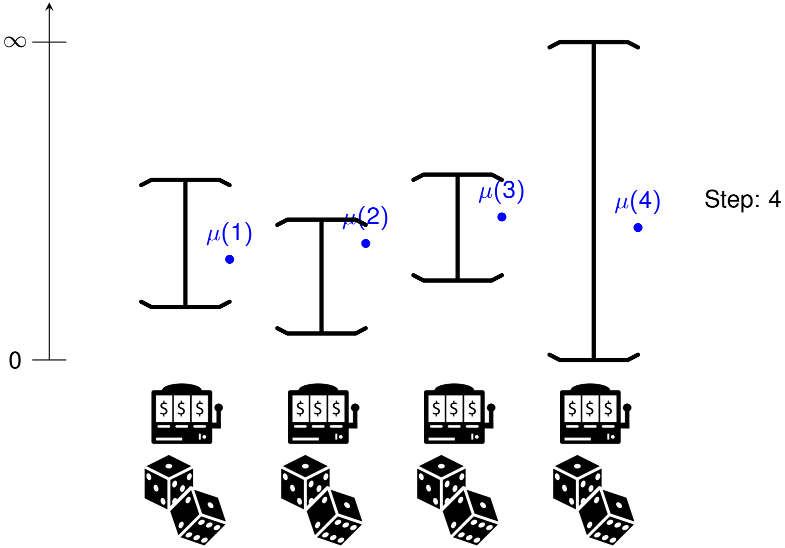

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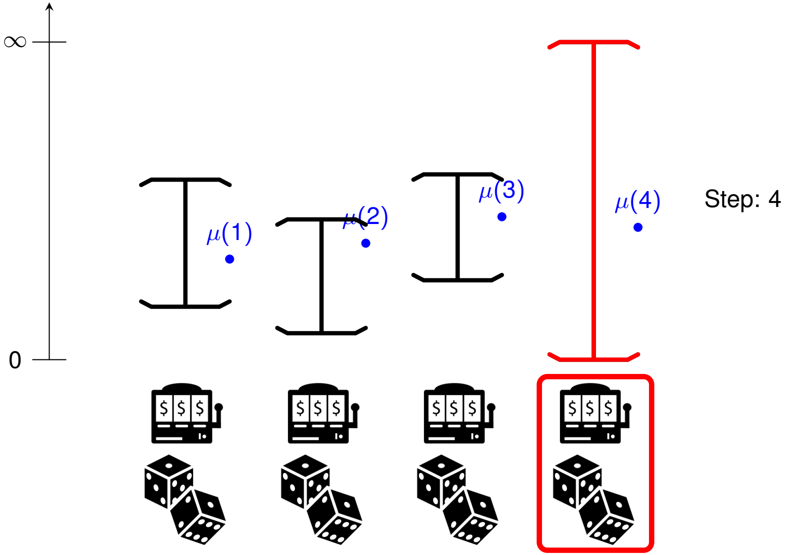

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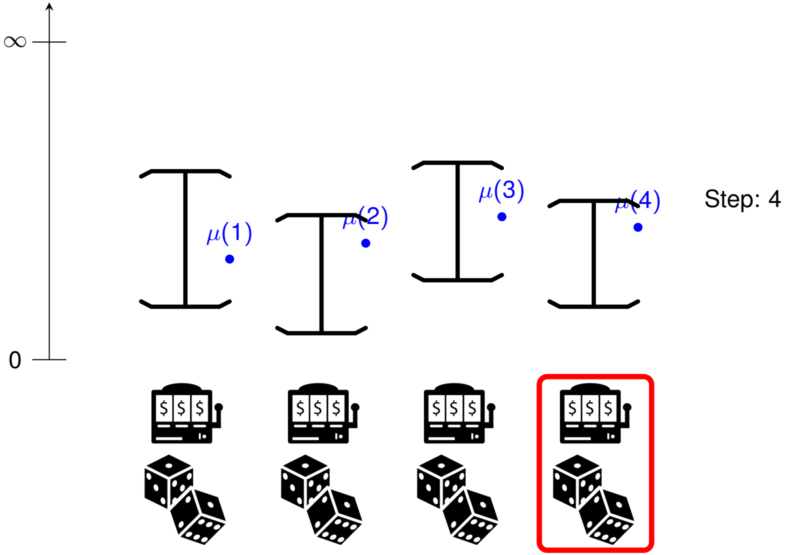

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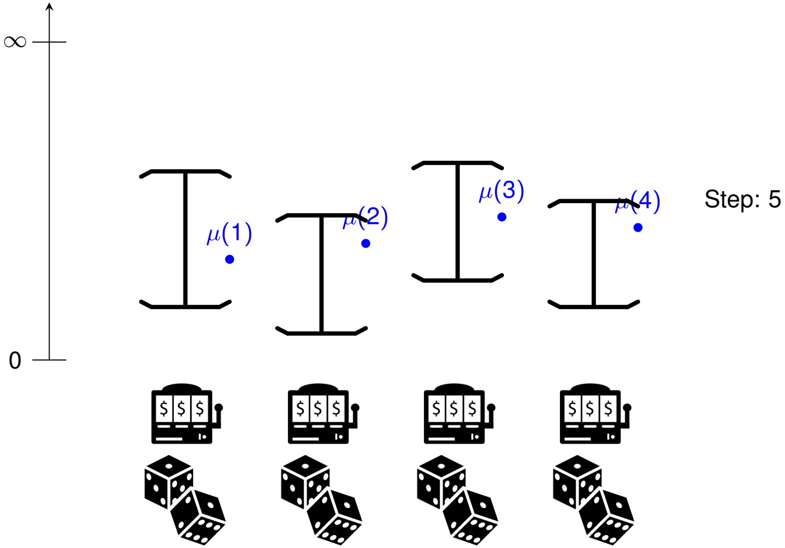

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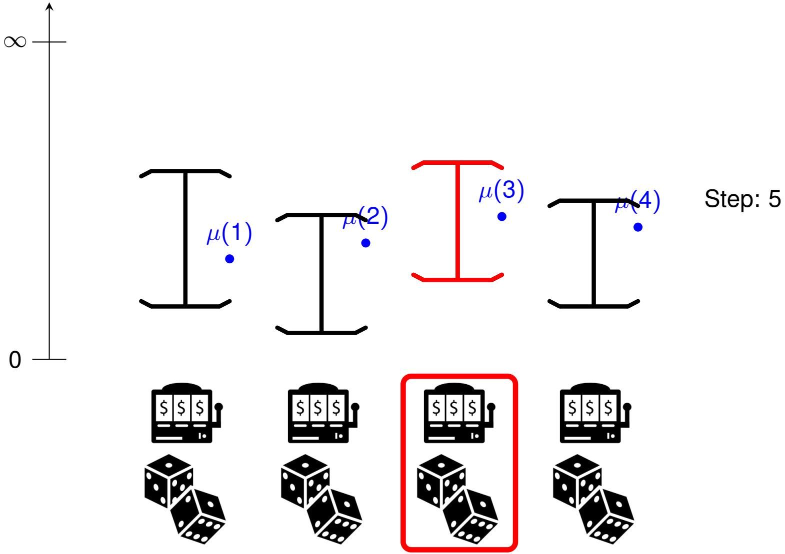

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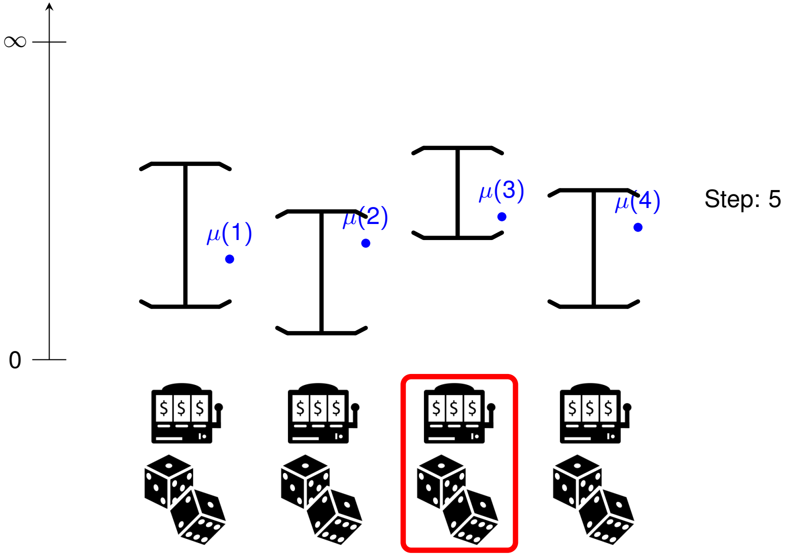

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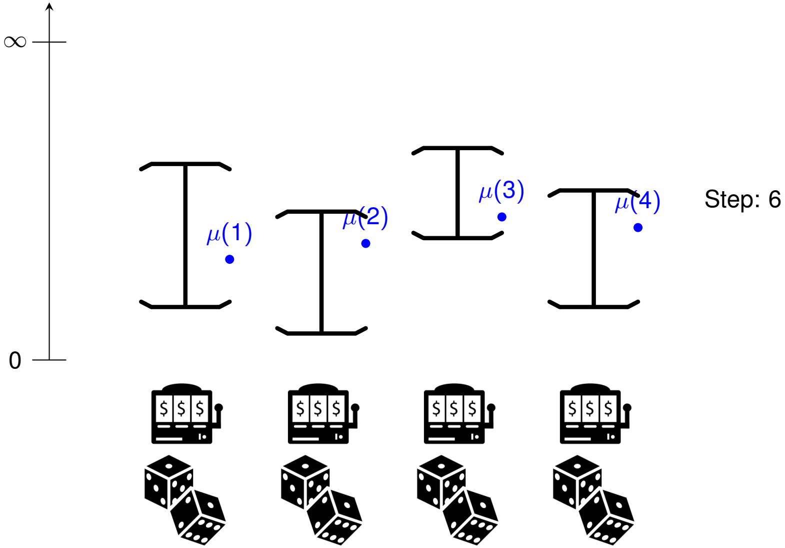

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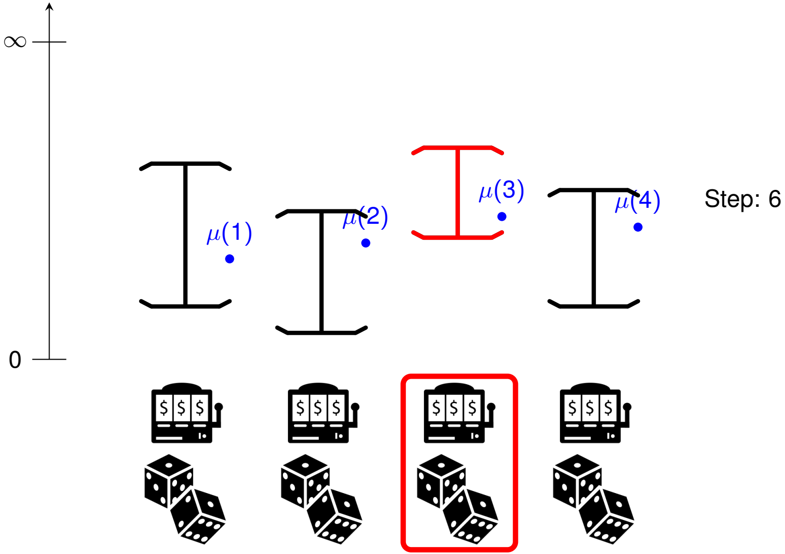

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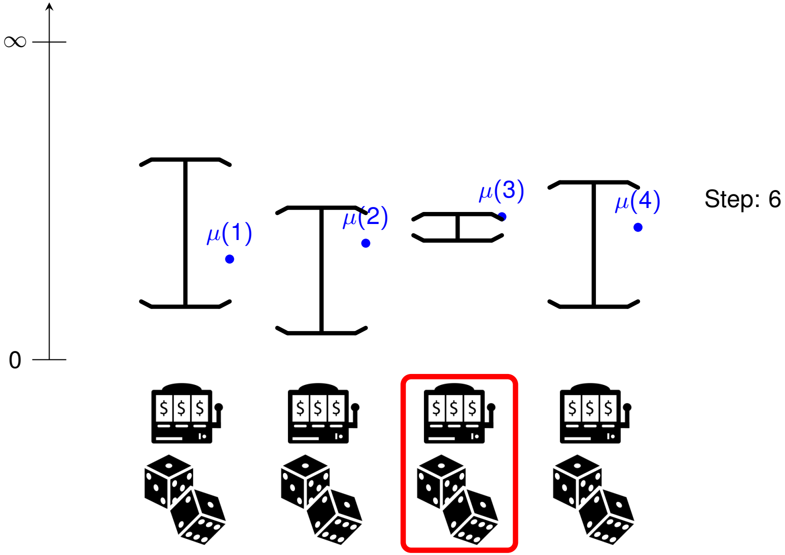

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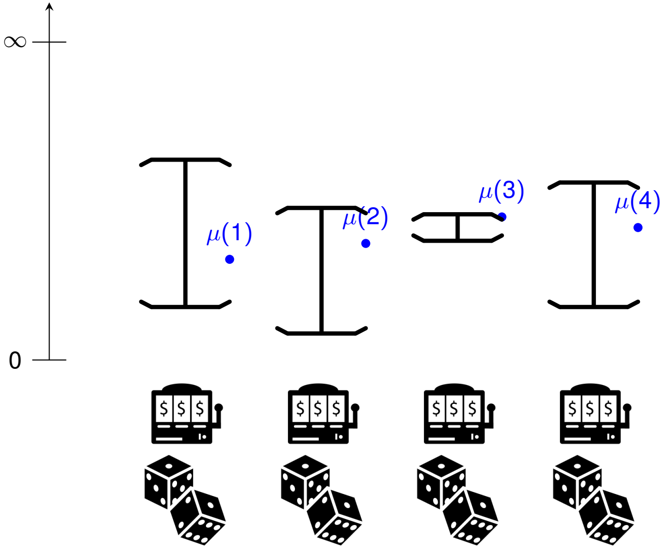

Step: 7

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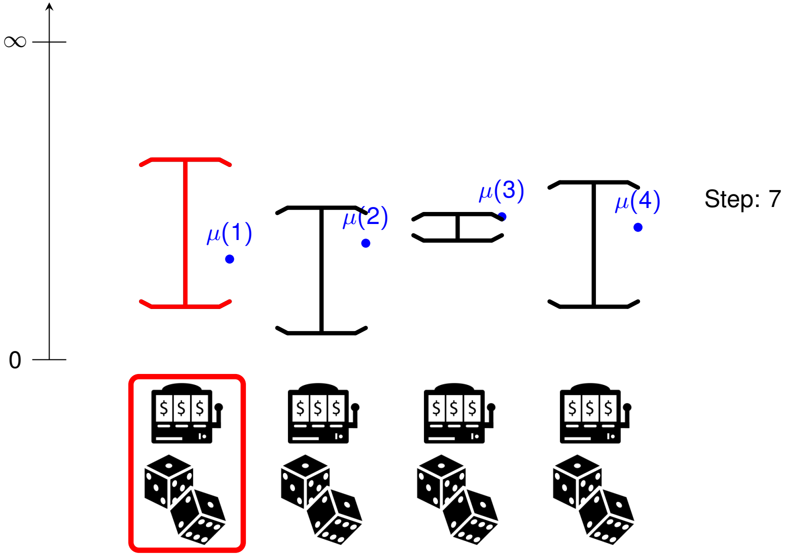

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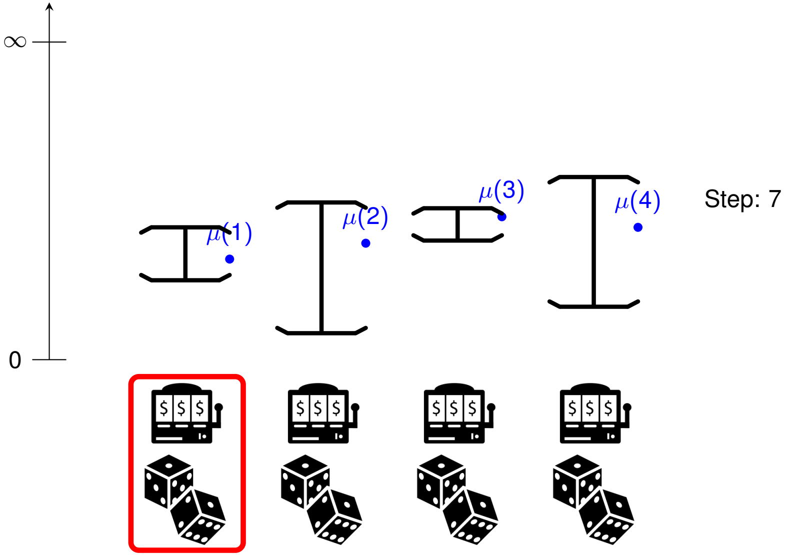

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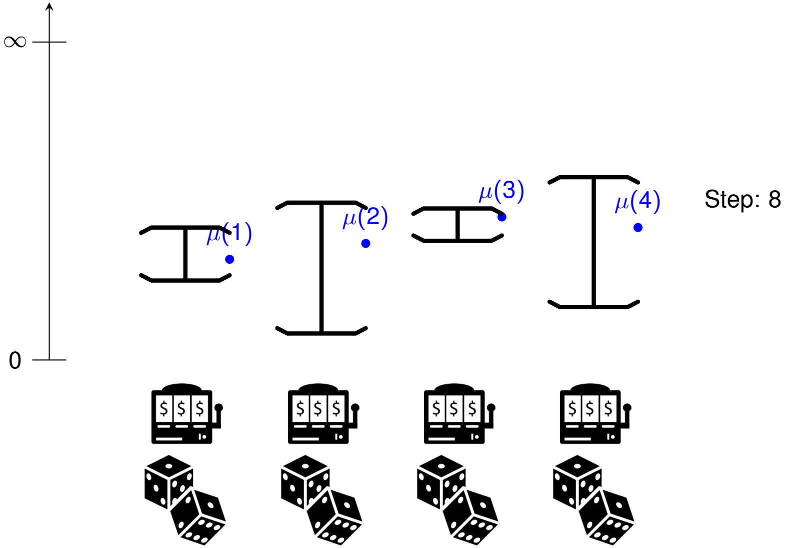

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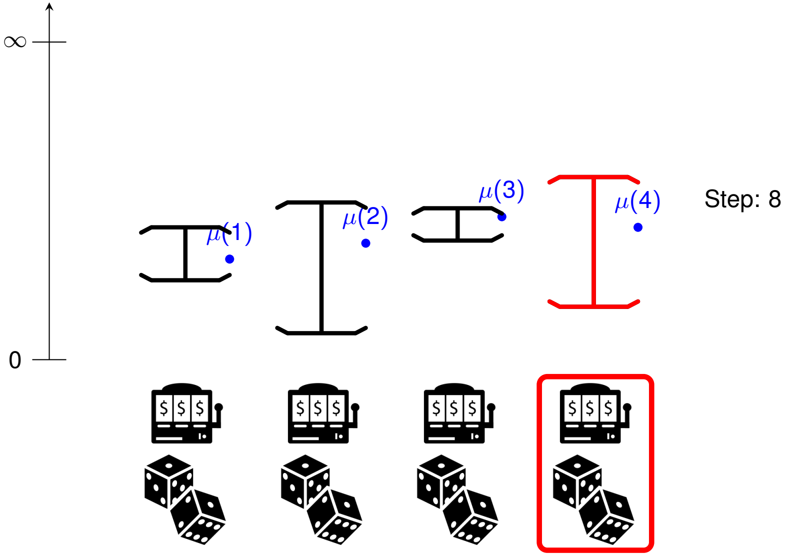

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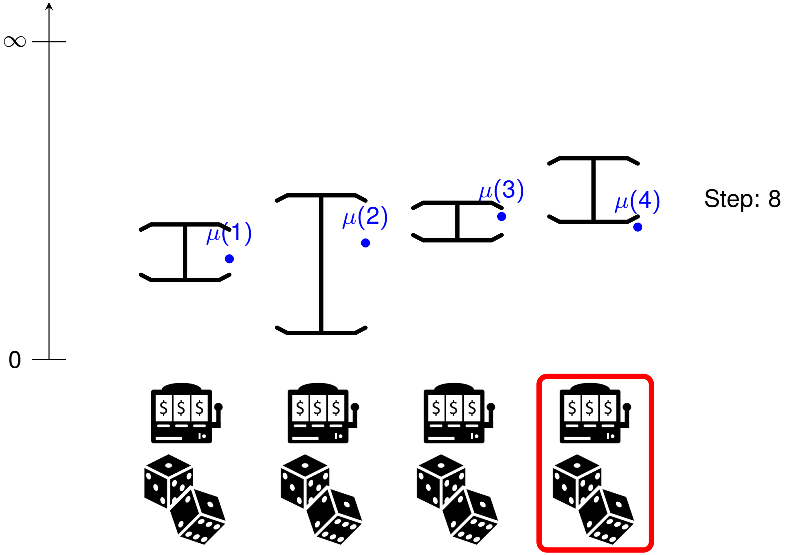

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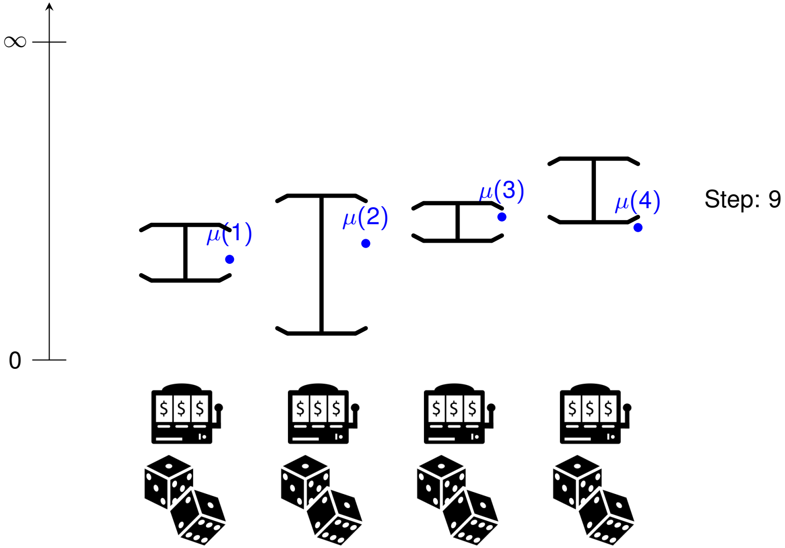

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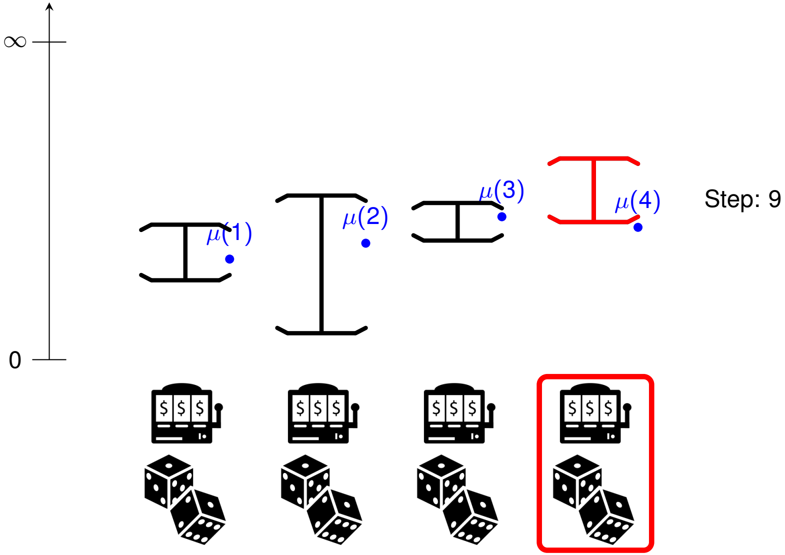

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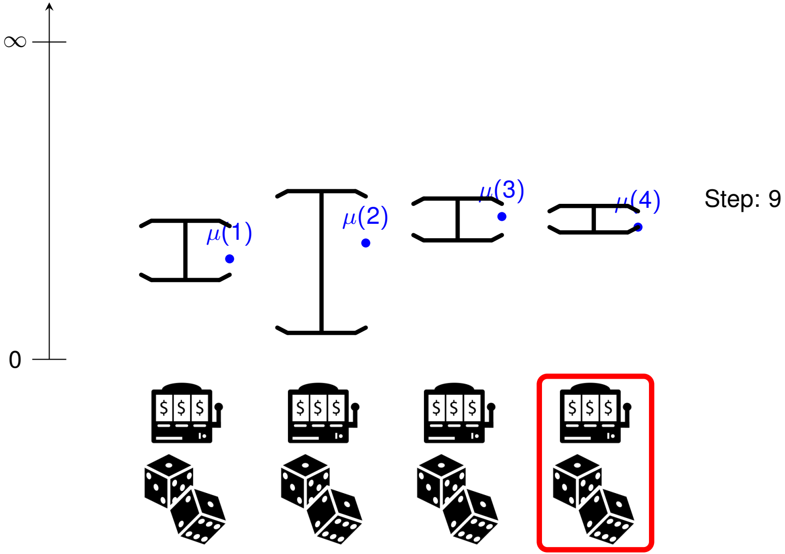

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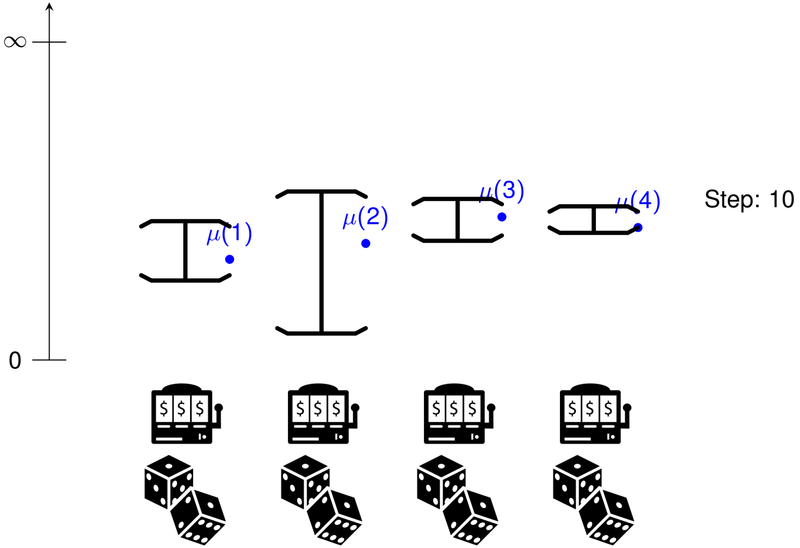

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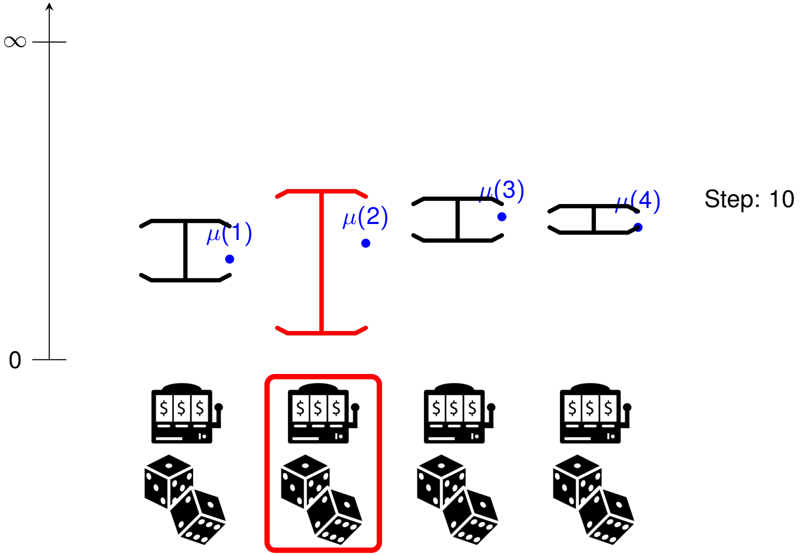
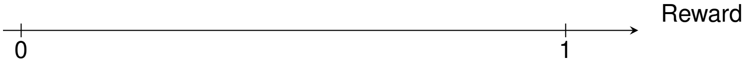


Intuition: How UCB avoids sub-optimal arms



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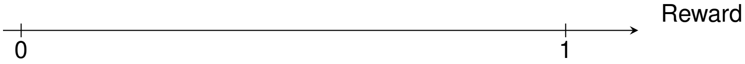

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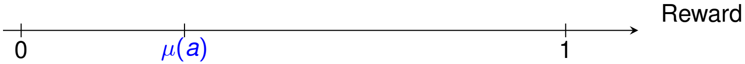

- Let a be a sub-optimal action

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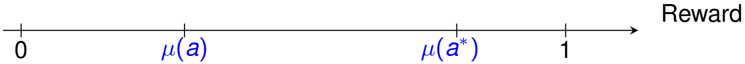

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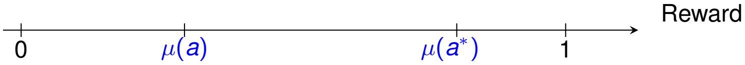

- Let a be a sub-optimal action with $\mu(a) \leq \mu(a^*) - \Delta$

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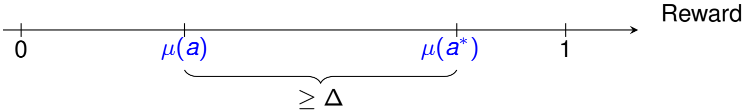

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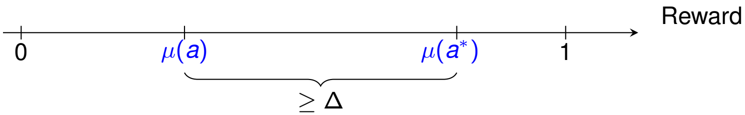

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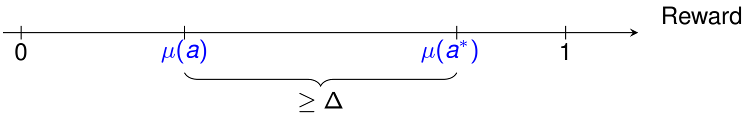

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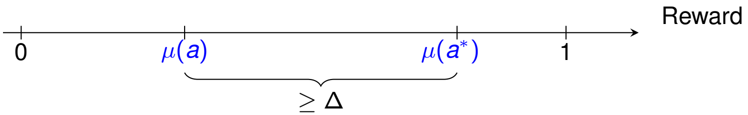

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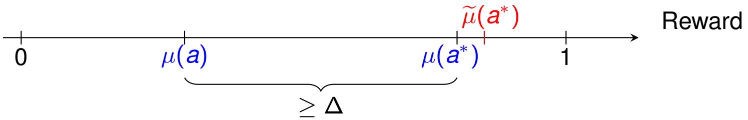

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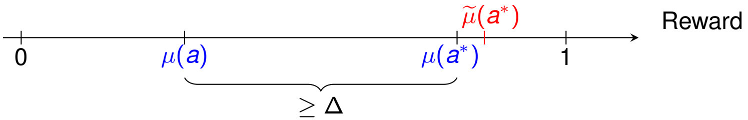

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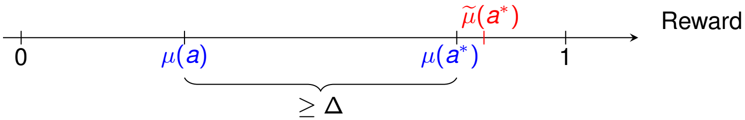

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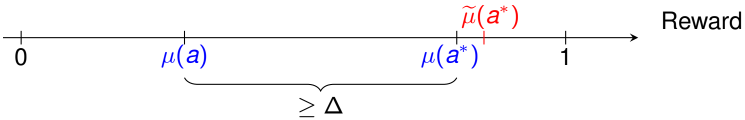

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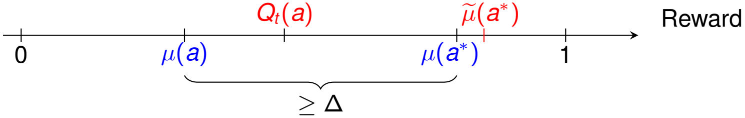

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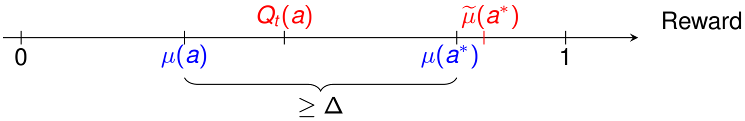

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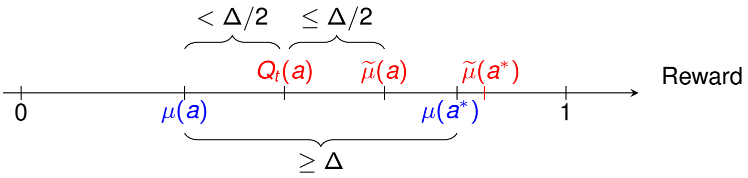

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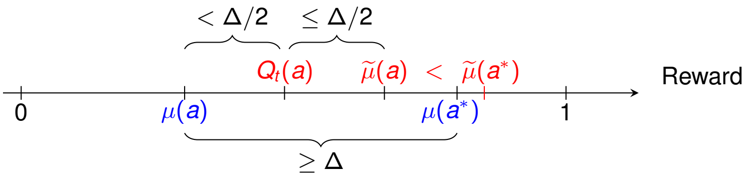

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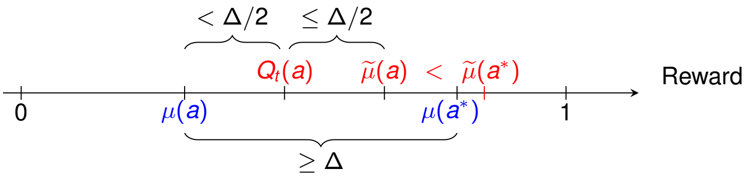

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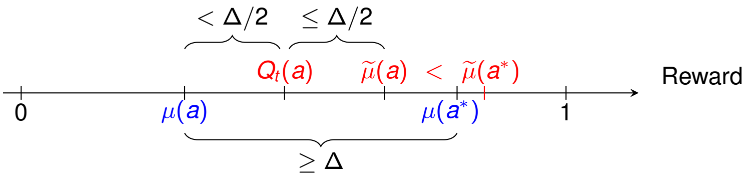

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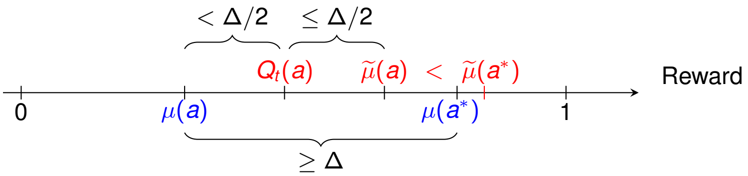

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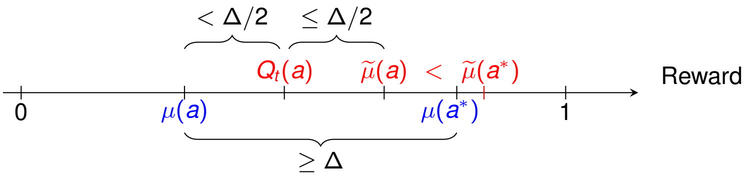

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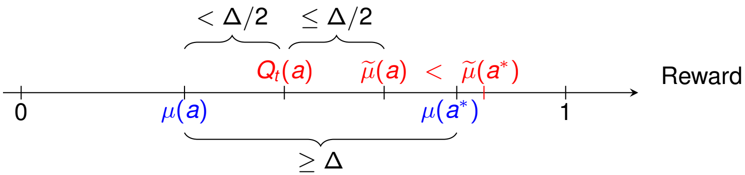

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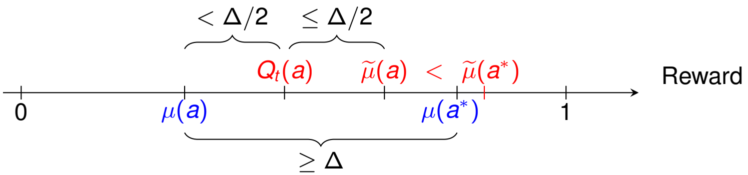

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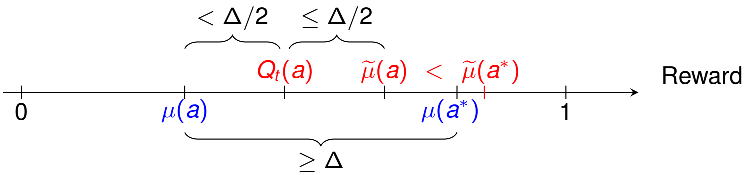

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One can also prove a lower bound of $\Omega(\log(T))$ for any algorithm!

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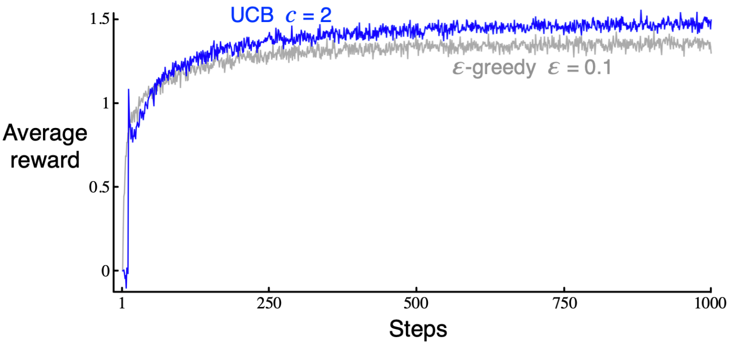
Experimental Results: ϵ-Greedy and UCB



Source: Sutton and Barto

Notes:

- This is the same bandit setting as on slides 20–21
- The UCB algorithm above uses $\Delta_t(a) = 2\sqrt{\frac{\log(t)}{n_t(a)}}$

Thank you and Best Wishes for the Exam!

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If you have any questions, comments or feedback, please send an email to `tms41@cam.ac.uk`

Outline

Introduction

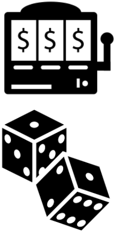
Stochastic Bandits

Outlook: Adversarial Bandits (non-examinable)

Why Adversarial Bandits?

Stochastic Bandits

- Rewards of each arm are i.i.d. samples in $[0, 1]$
- distribution is specific to each arm but is time-invariant (stationarity)



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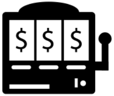

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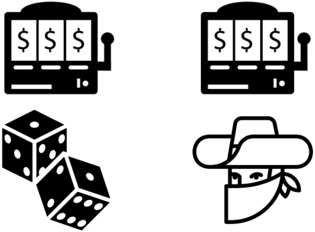

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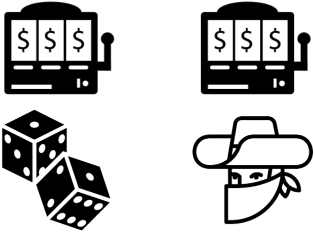

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Very weak assumptions $\rightsquigarrow$ powerful model!

Bandits with Full Information: Online Learning using Experts

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The Multiplicative Weights Algorithm (MWA)

Initialization: Fix $\delta \leq 1/2$. For every $i \in [n]$, let $w_i^{(1)} := 1$

Update: For $t = 1, 2, \ldots, T$:

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- Problem: In the bandit-setting, we only observe the cost (reward) of the **taken action**

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The EXP3-Algorithm

EXP3 = Exponential-weight algorithm for Exploration and Exploitation

The EXP3-Algorithm

Initialization: Fix $\gamma \in (0,1)$. Let $w_1(a) := 1$ for each of the k actions

For $t = 1, 2, \ldots, T$:

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$$p_t(a) := \frac{w_t(a)}{\sum_{a'} w_t(a')},$$

action is sampled proportional to weights!

and choose action i with probability $p_t(a)$.

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EXP3-algorithm tries to emulate the full information (expert) setting!

Analysis of EXP3-Algorithm

Performance of EXP3-Algorithm (Auer, Cesa-Bianchi, Freund, Shapire 2002)

For any $T \geq 1$, the expected regret of EXP3 with $\gamma = \sqrt{\frac{\log(k)}{kT}}$ satisfies

$$R_T \leq 2\sqrt{T \cdot k \log(k)}.$$

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In the full-information (expert setting), we could achieve $R_T = O(\sqrt{T \log(k)})$!

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