Randomised Algorithms

Lecture 14: Online Learning with Experts

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Outline

Introduction

Deterministic Weighted Majority

Randomised Weighted Majority

Extensions and Conclusions

Landscape of Machine Learning Algorithms

Training Set provided initially

Feedback after Decisions

No Training Set

Supervised Learning

Classification, regression: logistic regr., SVM, decision tree, neural networks, naive Bayes, Perceptron, kNN, Boosting

Online and Reinforcement Learning

Expert Learning: Weighted-Majority, Multiplicative-Update, Markov Decision Processes: Multi-Armed Bandits, Q-Learning

Unsupervised Learning

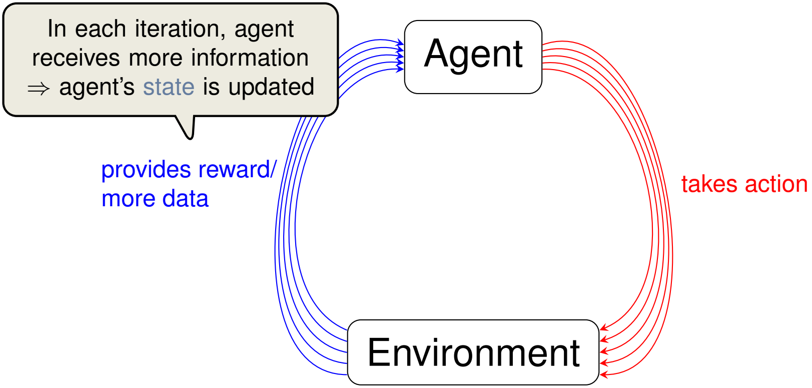
Clustering: spectral, hierarchical, k-means; Dimensionality Reduction, PCA, SVD

Predict unseen data

Maximise Reward

Extract Knowledge

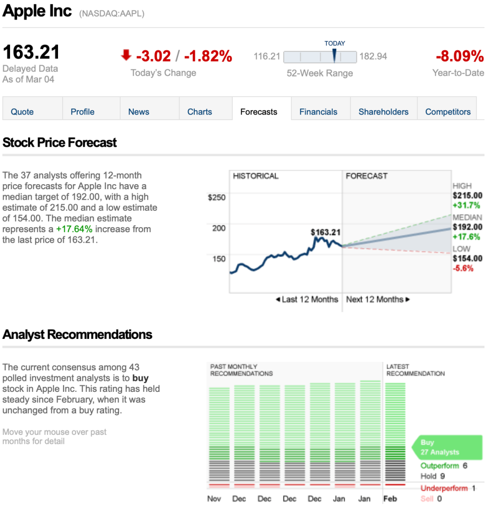
Online Algorithm/Reinforcement Learning Framework



Iteration: 4

Source: Yahoo Finance, 5 March 2022

Disclaimer: This is only given as a high-level motivation for the algorithm. It is not suggested to use any of the following ideas in practice at this or any other point.



Source: CNN Money, 5 March 2022

Other Applications: Spam Filtering, Weather Prediction, . . .

Outline

Online Learning using Expert Advice

Basic Setup

- Assume there is a single stock, and daily price movement is a sequence of binary events (up $= 1$ /down $= 0$)
- The stock movements can be arbitrary (i.e., adversarial)
- We are allowed to watch n experts (these might be arbitrarily bad and correlated)

Weighted Majority Algorithm

Initialization: Fix $\delta \leq 1/2$. For every $i \in [n]$, let $w_i^{(1)} := 1$

Update: For $t = 1, 2, \ldots, T$:

- Make prediction which is the weighted majority of the experts' predictions
- For every expert i who predicts wrongly, decrease his weight by a factor of $(1 - \delta)$:

$$w_i^{(t+1)} = (1 - \delta) w_i^{(t)}$$

Example of an **ensemble method**, combining advice from several other "algorithms".

Weighted Majority Algorithm: Example

Let $\delta = 1/2$, $n = 3$

t	Expert Weights	Expert Predictions	Our Pred.	Result	Our Errors
1	$1, 1, 1$	$1, 1, 0$	1 ✓	1	0
2	$1, 1, 1/2$	$0, 1, 0$	0 ✗	1	1
3	$1/2, 1, 1/4$	$1, 0, 1$	0 ✓	0	1
4	$1/4, 1, 1/8$	$0, 1, 1$	1 ✗	0	2
5	$1/4, 1/2, 1/16$	$1, 1, 0$	1 ✓	1	2
6	$1/4, 1/2, 1/32$	$0, 1, 1$	1 ✓	1	2
7	$1/8, 1/2, 1/32$	$0, 1, 0$	1 ✗	0	3
8	$1/8, 1/4, 1/32$	$1, 0, 1$	0 ✗	1	4
9	$1/8, 1/8, 1/32$	$0, 0, 0$	0 ✓	0	4
10	$1/8, 1/8, 1/32$	$1, 0, 1$	1 ✗	0	5
11	$1/16, 1/8, 1/64$	–	–	–	–

$\Rightarrow$ We made 5 mistakes, while the best expert made only 3 mistakes.
This looks quite bad, but the example is **too small** to draw conclusions!

Analysis of the Weighted Majority Algorithm

Notation: Let $m_i^{(t)}$ be the number of mistakes of expert i after t steps.

Analysis

The number of mistakes of our algorithm $M^{(T)}$ satisfies

$$M^{(T)} \leq 2 \cdot (1+\delta) \cdot \min_{i \in [n]} m_i^{(T)} + \frac{2 \ln n}{\delta}.$$

This bound holds for any input, any T and any δ!

Proof Outline:

- Define $\Phi^{(t)} = \sum_{i=1}^{n} w_i^{(t)}$ as the sum of weights
- Update Rule: If we make many mistakes, then $\Phi^{(t)}$ becomes small
- For $\Phi^{(t)}$ to be small, all weights must be small
  ($\Rightarrow$ even the best expert must make many mistakes)

Analysis of the Weighted Majority Algorithm

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This bound holds for any input, any T and any δ!

Proof:

- Define a potential function $\Phi^{(t)} = \sum_{i=1}^{n} w_i^{(t)}$, so that $\Phi^{(1)} = n$.
- By induction, $w_i^{(t+1)} = (1-\delta)^{m_i^{(t)}}$ (see example!)
  - **Case 1:** Each time we are wrong, the weighted majority of experts is wrong $\Rightarrow$ at least half the total weight decreases by $1-\delta$:

$$\Phi^{(t+1)} \leq \Phi^{(t)} \cdot \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1-\delta) \right) = \Phi^{(t)} \cdot (1 - \delta/2).$$

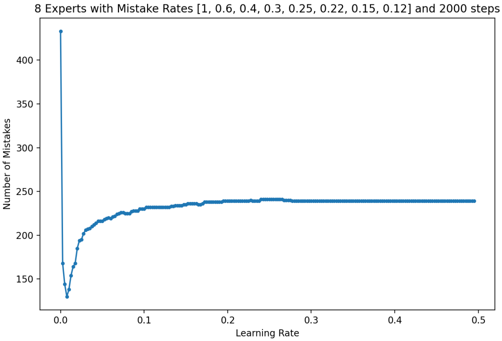
  - **Case 2:** Each time we are correct, $\Phi^{t+1} \leq \Phi^{t}$.
- By induction, $\Phi^{(T+1)} \leq n \cdot (1-\delta/2)^{M^{(T)}}$, but also $\Phi^{(T+1)} \geq w_i^{(T+1)} = (1-\delta)^{m_i^T}$.
- Taking logs:

$$m_i^{(T)} \ln(1-\delta) \leq M^{(T)} \ln(1-\delta/2) + \ln(n).$$

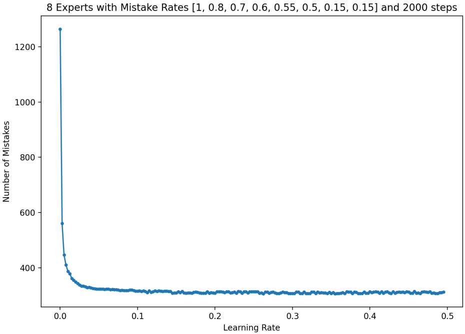
- Using now that $-\delta \geq \ln(1-\delta) \geq -\delta - \delta^2$ completes the proof. □

Simulation of the (Deterministic) WMA (1/2)

- Probabilistic Setting: Each expert i predicts wrongly with some probability $p_i \in [0, 1]$, independently across rounds and experts
- Question: Which learning rate works best?



Simulation of the (Deterministic) WMA (2/2)



Observations from these Experiments

- Depending on data set, a high or small learning rate may work best
- **But:** for such a random environment, other Machine Learning techniques (e.g., Naive Bayes or Neural Networks) work much better

The point of WMA is a strong **worst-case** guarantee!

Improving the Weighted Majority Algorithm?

Analysis

The number of mistakes of our algorithm $M^{(T)}$ satisfies

$$M^{(T)} \leq 2 \cdot (1+\delta) \cdot \min_{i \in [n]} m_i^{(T)} + \frac{2 \ln n}{\delta}.$$

Question: Is there a way to avoid the factor of 2?

Exercise: For any **deterministic** algorithm, the factor of 2 cannot be avoided!

Idea: Employ a randomised strategy which selects an expert with probability proportional to its success!

Outline

Randomised Weighted Majority

Randomised Weighted Majority Algorithm

Initialization: Fix $\delta \leq 1/2$. For every $i \in [n]$, let $w_i^{(1)} := 1$

Update: For $t = 1, 2, \ldots, T$:

- Pick expert i with probability proportional to w_i and follow that prediction
- For every expert i who predicts wrongly, decrease his weight by a factor of $(1 - \delta)$:

$$w_i^{(t+1)} = (1 - \delta) w_i^{(t)}$$

Note that the number of mistakes we are making is now a random variable!

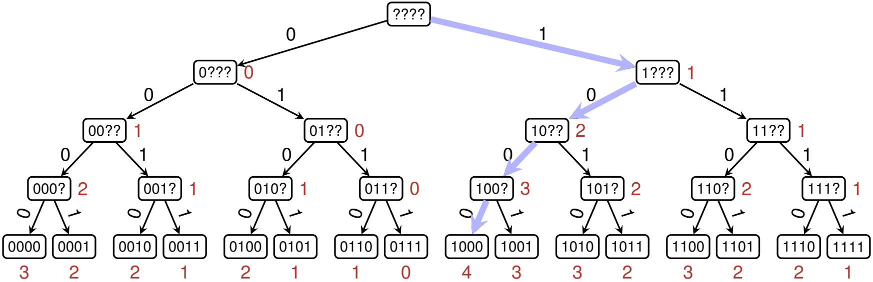
Example: Deterministic vs. Randomised Weighted Majority (1/2)

Consider the following run of the Deterministic Weighted Majority Algorithm:

t	Weights	Predictions	Our Prediction	Actual Result	Our Errors
1	1,1	1,0	1	0	1
2	1/2,1	1,0	0	1	2
3	1/2,1/2	0,1	0	1	3
4	1/4,1/2	1,0	0	1	4
5	1/4,1/4	–	–	–	–

Consider now the Randomised Weighted Majority Algorithm and let us compute the expected number of mistakes, $\mathbf{E}\left[M^{(4)}\right]$

Example: Deterministic vs. Randomised Weighted Majority (2/2)



- Let $x^{(t)}$ be a $0/1$ random variable, indicating if our t-th prediction is wrong.
- Then:

$$\mathbf{E}\left[x^{(1)}\right] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

- Similarly, $\mathbf{E}\left[x^{(2)}\right] = \frac{2}{3}$, $\mathbf{E}\left[x^{(3)}\right] = \frac{1}{2}$ and $\mathbf{E}\left[x^{(4)}\right] = \frac{2}{3}$.
- Hence,

$$\mathbf{E}\left[M^{(4)}\right] = \mathbf{E}\left[x^{(1)} + x^{(2)} + x^{(3)} + x^{(4)}\right]$$

$$= \mathbf{E}\left[x^{(1)}\right] + \mathbf{E}\left[x^{(2)}\right] + \mathbf{E}\left[x^{(3)}\right] + \mathbf{E}\left[x^{(4)}\right] = \frac{7}{3}$$

Much better than the deterministic algorithm!

Analysis of Randomised Weighted Majority (non-examinable)

Analysis

The expected number of mistakes of our algorithm $M^{(T)}$ satisfies

$$\mathbf{E}\left[M^{(T)}\right] \leq 1 \cdot (1+\delta) \cdot \min_{i \in [n]} m_i^{(T)} + \frac{\ln n}{\delta}.$$

This was a factor of 2 before!

Proof:

- Define a potential function $\Phi^{(t)} = \sum_{i=1}^n w_i^{(t)}$, so that $\Phi^{(1)} = n$.
- The probability of picking expert i in round t is $p_i^{(t)} := w_i^{(t)} / \sum_{j=1}^n w_j^{(t)} = w_i^{(t)} / \Phi^{(t)}$.
- Let $\lambda_i^{(t)}$ be 1 iff expert i is wrong at time t (and 0 otherwise)
- Then the expected number of mistakes by our algorithm is $\mathbf{E}[M^{(T)}] = \sum_{t=1}^T \lambda^{(t)} \cdot p^{(t)}$.
- The new potential (which is deterministic!) can be upper bounded by:

$$\Phi^{(t+1)} = \sum_{i=1}^n w_i^{(t+1)} = \sum_{i=1}^n (1-\delta\lambda_i^{(t)}) \cdot w_i^{(t+1)} = \Phi^{(t)} \cdot \left(1 - \delta\lambda^{(t)} p^{(t)}\right) \leq \Phi^{(t)} \cdot \exp\left(-\delta\lambda^{(t)} \cdot p^{(t)}\right)$$

- Thus the final potential satisfies

$$\Phi^{(T+1)} \leq \Phi^{(1)} \cdot \exp\left(-\delta \sum_{t=1}^T \lambda^{(t)} \cdot p^{(t)}\right) = n \cdot \exp\left(-\delta \cdot \mathbf{E}\left[M^{(T)}\right]\right),$$

$$\Phi^{(T+1)} \geq w_i^{(T+1)} = \prod_{t=1}^T \left(1 - \delta\lambda_i^{(t)}\right) = (1-\delta)^{m_i^{(T)}}$$

$\ln(1-\delta) \geq -\delta - \delta^2$

$$\Rightarrow \quad \ln n - \delta \cdot \mathbf{E}[M^{(T)}] \geq \ln(1-\delta) \cdot m_i^{(T)} \quad \Rightarrow \quad \mathbf{E}[M^{(T)}] \leq \frac{(\delta+\delta^2)}{\delta} \cdot m_i^{(T)} + \frac{\ln n}{\delta} \qquad \square$$

Optimising the Learning Rate

Analysis

The expected number of mistakes of our algorithm $M^{(T)}$ satisfies

$$\mathbf{E}\left[M^{(T)}\right] \leq 1 \cdot (1+\delta) \cdot \min_{i \in [n]} m_i^{(T)} + \frac{\ln n}{\delta}.$$

Interpretation:

- Suppose that T is known in advance
- Pick learning rate $\delta = \sqrt{\ln(n)/T}$
  (assuming T is large enough so that $\delta \leq 1/2$!)

$$\mathbf{E}\left[M^{(T)}\right] \leq \min_{i \in [n]} m_i^{(T)} + \sqrt{\ln(n)/T} \cdot T + \sqrt{\ln(n) \cdot T}$$

$$= \min_{i \in [n]} m_i^{(T)} + 2 \cdot \sqrt{T \ln(n)}$$

Additive error (“regret”) negligible in most cases compared to $\min_{i \in [n]} m_i^{(T)}$!

Can we do better than that?

A Tight Lower Bound

Corollary

For $\delta = \sqrt{\ln(n)/T}$, the expected number of our mistakes $M^{(T)}$ satisfies

$$\mathbf{E}\left[M^{(T)} \right] \leq \min_{i\in[n]} m_i^{(T)} + 2\cdot\sqrt{T\ln(n)}.$$

- Suppose every expert $i = 1, 2, \ldots, n$ flips an unbiased coin, and the result is also an unbiased coin flip (independent of the experts' predictions)
- $\Rightarrow$ Regardless of our algorithm, the number of our mistakes satisfies

$$\mathbf{E}\left[M^{(T)} \right] = T \cdot \frac{1}{2}$$

- How good is the best expert?
  - Every expert $i \in [n]$ will make $T/2 \pm \Theta(\sqrt{T})$ many mistakes
  - Best expert will make $T/2 - \Theta(\sqrt{T\ln(n)})$ many mistakes (proof omitted, is based on central limit theorem)

- This demonstrates tightness of the error term
- Best expert will be good just by chance!

Outline

Choosing the Learning Rate Dynamically (1/2)

Question: How to adjust learning rate if T is not known in advance?

Approach 1: "The Doubling Trick"

- Algorithm:
  1. For $m = 1, 2, \ldots$
  2. Run a new instance of algorithm on the 2^m rounds $t = 2^m, \ldots, 2^{m+1} - 1$ with "optimal" learning rate (for an algorithm that runs for 2^m steps)
- Analysis before shows that in phase m, number of additional mistakes compared to best expert (regret) is at most $\alpha \cdot \sqrt{2^m}$

$\Rightarrow$ total regret after T steps is at most

$$\frac{\sqrt{2}}{\sqrt{2}-1}\alpha\sqrt{T},$$

where $\alpha = 2 \cdot \sqrt{\ln(n)}$.

Choosing the Learning Rate Dynamically (2/2)

Approach 2

- Algorithm:
  1. Run the randomised WMA with learning rate $\delta_t \approx 1/\sqrt{t}$ in round t
- A modification of our analysis proves for any time-interval $[T/2, T]$:

$$\sum_{t=T/2}^{T} \delta_t \cdot \lambda^{(t)} \cdot p^{(t)} \leq \log(\Phi^{(T/2)}) + (1-\delta_T)^{m_i^{[T/2,T]}}$$

$$\Rightarrow \qquad \mathbf{E}\left[M^{[T/2,T]} \right] \leq \frac{m_i^{[T/2,T]} \cdot \log(1-\delta_T)}{\delta_{T/2}} + \frac{\log(\Phi^{(T/2)})}{\delta_{T/2}}$$

Approach 3: "Self-Confident Algorithm"

- Algorithm:
  1. Run the randomised WMA with learning rate $\delta_t \approx 1/\sqrt{\min_{i\in[n]} m_i^{(t)}}$ (or $1/\sqrt{M^{(t)}}$) in round t

A More General Setting

New Setup

- At each step, we pick one expert i randomly out of n experts
- That expert i and our algorithm incur a cost $m_i^{(t)}$, but we also observe the costs of all experts (a vector $(m_j^{(t)})_{i=1}^n$)
- costs $m_j^{(t)}$ can be arbitrary in the range $[-1, 1]$

Coming back to our example of **stock prediction**:

- could define cost $m_j^{(t)} = 0$ if expert j is neutral (HOLD)
- cost $m_j^{(t)} > 0$ if expert j makes the wrong prediction (closer to 1 the stronger prediction and stronger the price change)
- cost $m_j^{(t)} < 0$ if expert j makes the correct prediction

Idea of the "Multiplicative Weights-Algorithm"

- In the first iteration, simply pick an expert uniformly at random
- Every expert will be penalised or rewarded through a multiplicative weight-update

The Multiplicative Weights Algorithm

The Multiplicative Weights Algorithm

Initialization: Fix $\delta \leq 1/2$. For every $i \in [n]$, let $w_i^{(1)} := 1$

Update: For $t = 1, 2, \ldots, T$:

- Choose expert i with prop. proportional to $w_i^{(t)}$.
- Observe the costs of all n experts in round t, $m^{(t)}$
- For every expert i, update its weight by:

$$w_i^{(t+1)} = (1 - \delta m_i^{(t)}) w_i^{(t)}$$

Analysis

For any expert i, the expected cost of this algorithm is at most

$$\sum_{t=1}^{T} m_i^{(t)} + \delta \cdot \sum_{t=1}^{T} \left| m_i^{(t)} \right| + \frac{\log n}{\delta}.$$

Derivation is very similar to the ones shown before.

Conclusions

Summary

- Weighted Majority Algorithm
  - natural, simple (and deterministic) algorithm
  - good performance, but could be a factor of 2 worse than the best expert
- Randomised Weighted Majority Algorithm
  - Randomised extension
  - almost optimal performance thanks to randomisation which guards against tailored worst-case instances (cmp. Quick-Sort!)
  - impact of the learning rate: small learning rate gives very good performance guarantees. However, actual performance may depend on the specific data set at hand (cf. simulations!)
- Multiplicative Weight-Update Algorithm
  - further generalisation of the (randomised) weighted majority algorithm

Outlook

- These algorithms are examples of the Ensemble-Method: Framework combining weak predictions into a strong learner
- A closely related algorithmic approach: Follow the Perturbed Leader
- Similar update schemes are Perceptron and AdaBoost

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