Randomised Algorithms

Lecture 14: Online Learning with Experts

Thomas Sauerwald (tms41@cam.ac.uk)

Lent 2022

Outline

Introduction

Deterministic Weighted Majority

Randomised Weighted Majority

Extensions and Conclusions

Landscape of Machine Learning Algorithms

Training Set provided initially

Feedback after Decisions

No Training Set

Supervised Learning

Classification, regression: logistic regr., SVM, decision tree, neural networks, naive Bayes, Perceptron, kNN, Boosting

Online and Reinforcement Learning

Expert Learning: Weighted-Majority, Multiplicative-Update, Markov Decision Processes: Multi-Armed Bandits, Q-Learning

Unsupervised Learning

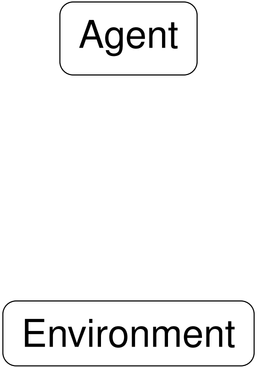
Clustering: spectral, hierarchical, k-means; Dimensionality Reduction, PCA, SVD

Predict unseen data

Maximise Reward

Extract Knowledge

Online Algorithm/Reinforcement Learning Framework



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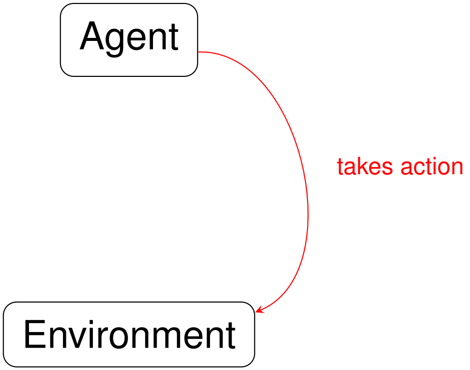

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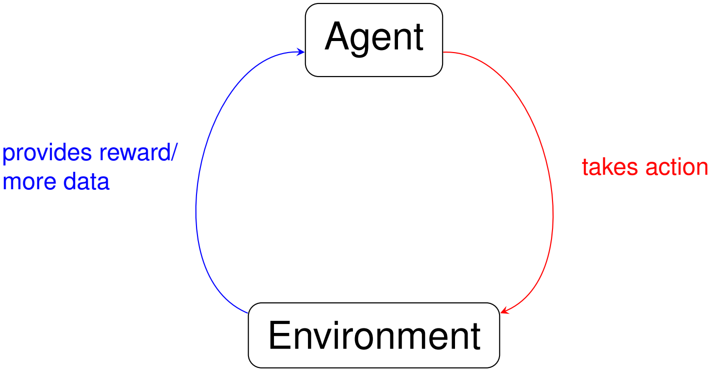

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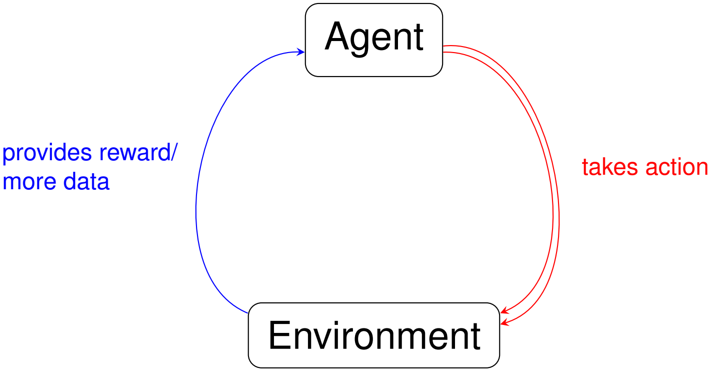

Iteration: 2

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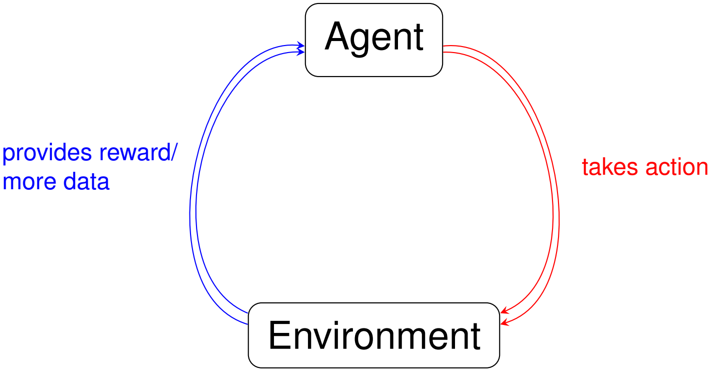

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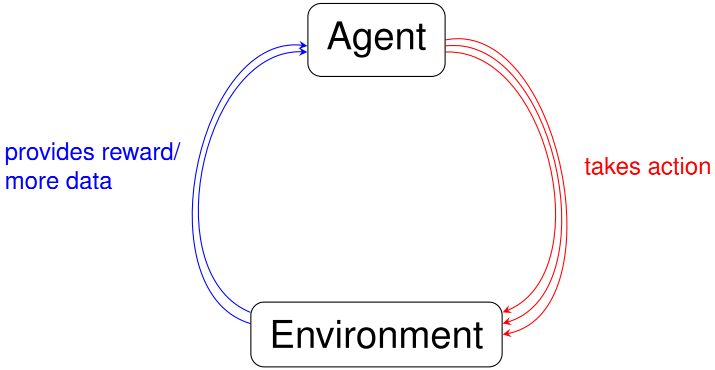

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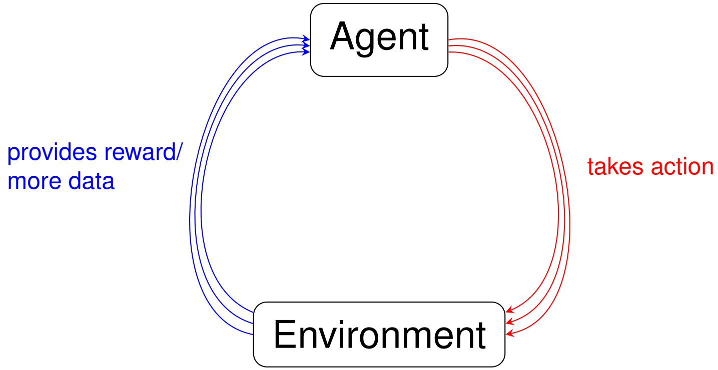

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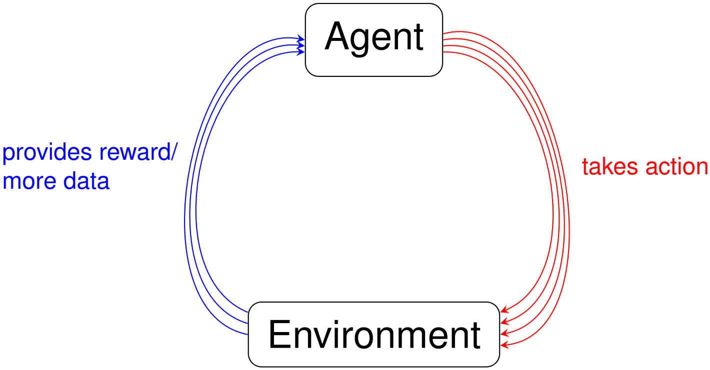

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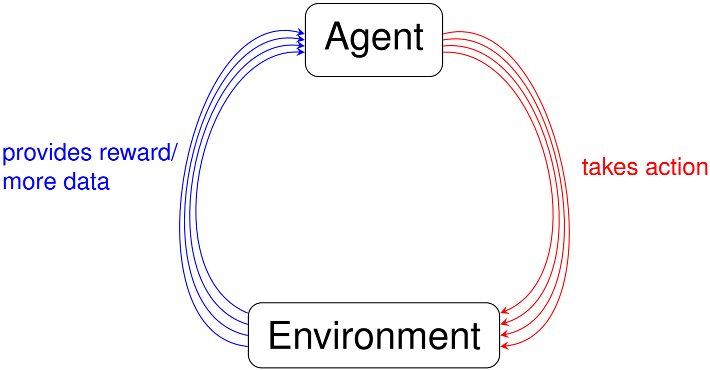

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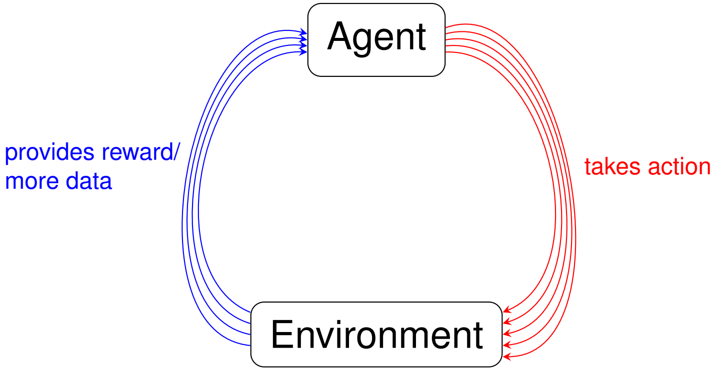

Iteration: 5

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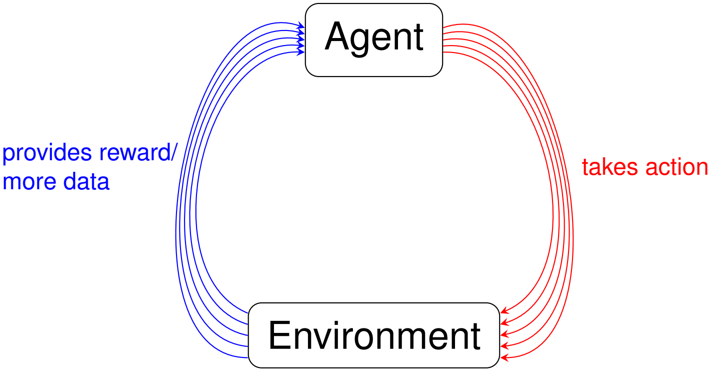

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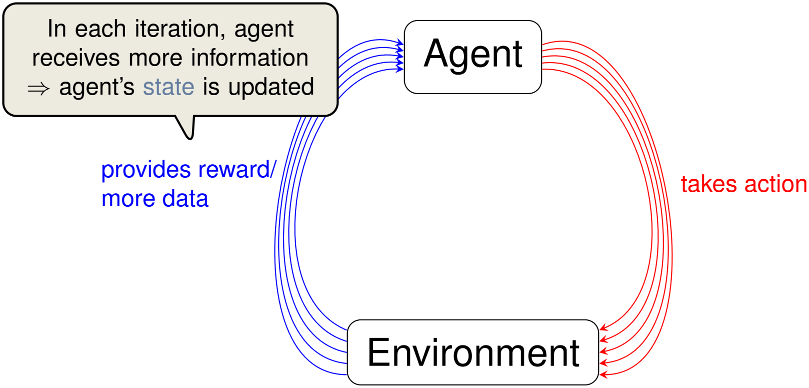

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Source: Yahoo Finance, 5 March 2022

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Disclaimer: This is only given as a high-level motivation for the algorithm. It is not suggested to use any of the following ideas in practice at this or any other point.

Apple Inc (NASDAQ:AAPL)

163.21
Delayed Data
As of Mar 04

-3.02 / -1.82%
Today's Change

TODAY
116.21 182.94
52-Week Range

-8.09%
Year-to-Date

Quote | Profile | News | Charts | Forecasts | Financials | Shareholders | Competitors

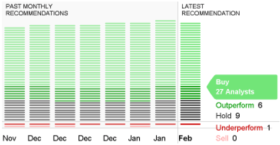
Stock Price Forecast

The 37 analysts offering 12-month price forecasts for Apple Inc have a median target of 192.00, with a high estimate of 215.00 and a low estimate of 154.00. The median estimate represents a +17.64% increase from the last price of 163.21.

Analyst Recommendations

The current consensus among 43 polled investment analysts is to **buy** stock in Apple Inc. This rating has held steady since February, when it was unchanged from a buy rating.

Move your mouse over past months for detail



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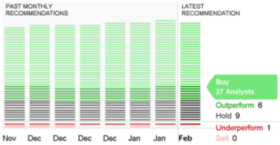

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Other Applications: Spam Filtering, Weather Prediction, . . .

Outline

Online Learning using Expert Advice

Basic Setup

- Assume there is a single stock, and daily price movement is a sequence of binary events (up $= 1$ /down $= 0$)
- The stock movements can be arbitrary (i.e., adversarial)
- We are allowed to watch n experts (these might be arbitrarily bad and correlated)

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Weighted Majority Algorithm

Initialization: Fix $\delta \leq 1/2$. For every $i \in [n]$, let $w_i^{(1)} := 1$

Update: For $t = 1, 2, \ldots, T$:

- Make prediction which is the weighted majority of the experts' predictions
- For every expert i who predicts wrongly, decrease his weight by a factor of $(1 - \delta)$:

$$w_i^{(t+1)} = (1 - \delta) w_i^{(t)}$$

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Example of an **ensemble method**, combining advice from several other "algorithms".

Weighted Majority Algorithm: Example

Let $\delta = 1/2$, $n = 3$

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$\Rightarrow$ We made 5 mistakes, while the best expert made only 3 mistakes.
This looks quite bad, but the example is **too small** to draw conclusions!

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Notation: Let $m_i^{(t)}$ be the number of mistakes of expert i after t steps.

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The number of mistakes of our algorithm $M^{(T)}$ satisfies

$$M^{(T)} \leq 2 \cdot (1+\delta) \cdot \min_{i \in [n]} m_i^{(T)} + \frac{2 \ln n}{\delta}.$$

This bound holds for any input, any T and any δ!

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Proof Outline:

- Define $\Phi^{(t)} = \sum_{i=1}^{n} w_i^{(t)}$ as the sum of weights

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The number of mistakes of our algorithm $M^{(T)}$ satisfies

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- Define $\Phi^{(t)} = \sum_{i=1}^{n} w_i^{(t)}$ as the sum of weights
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  ($\Rightarrow$ even the best expert must make many mistakes)

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- Using now that $-\delta \geq \ln(1-\delta) \geq -\delta - \delta^2$ completes the proof. $\square$

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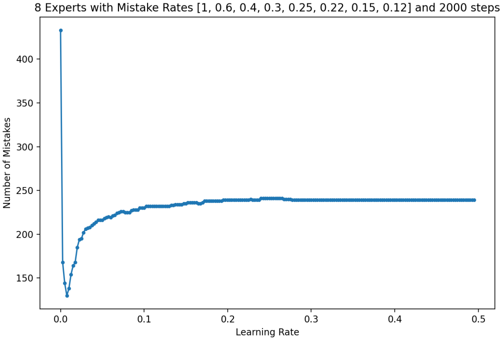

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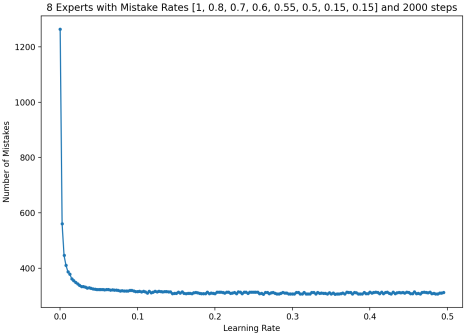

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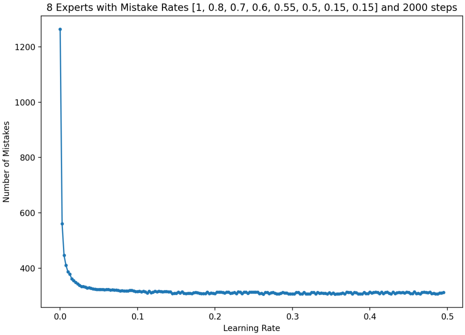

Observations from these Experiments

- Depending on data set, a high or small learning rate may work best
- **But:** for such a random environment, other Machine Learning techniques (e.g., Naive Bayes or Neural Networks) work much better

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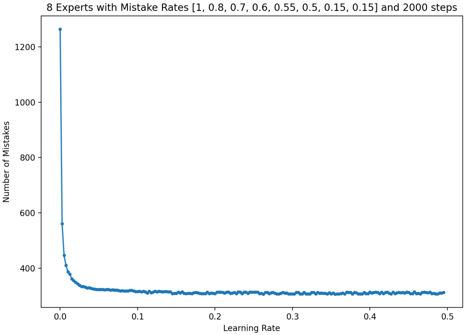

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The point of WMA is a strong **worst-case** guarantee!

Improving the Weighted Majority Algorithm?

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Question: Is there a way to avoid the factor of 2?

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Idea: Employ a randomised strategy which selects an expert with probability proportional to its success!

Outline

Introduction

Deterministic Weighted Majority

Randomised Weighted Majority

Extensions and Conclusions

Randomised Weighted Majority Algorithm

Initialization: Fix $\delta \leq 1/2$. For every $i \in [n]$, let $w_i^{(1)} := 1$

Update: For $t = 1, 2, \ldots, T$:

- Pick expert i with probability proportional to w_i and follow that prediction
- For every expert i who predicts wrongly, decrease his weight by a factor of $(1 - \delta)$:

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Note that the number of mistakes we are making is now a random variable!

Example: Deterministic vs. Randomised Weighted Majority (1/2)

Consider the following run of the Deterministic Weighted Majority Algorithm:

t	Weights	Predictions	Our Prediction	Actual Result	Our Errors
1	1,1	1,0	1	0	1
2	1/2,1	1,0	0	1	2
3	1/2,1/2	0,1	0	1	3
4	1/4,1/2	1,0	0	1	4
5	1/4,1/4	–	–	–	–

Consider now the Randomised Weighted Majority Algorithm and let us compute the expected number of mistakes, $\mathbf{E}\left[M^{(4)} \right]$

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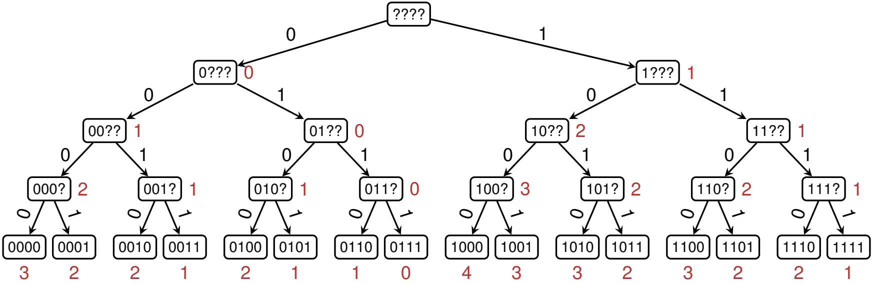

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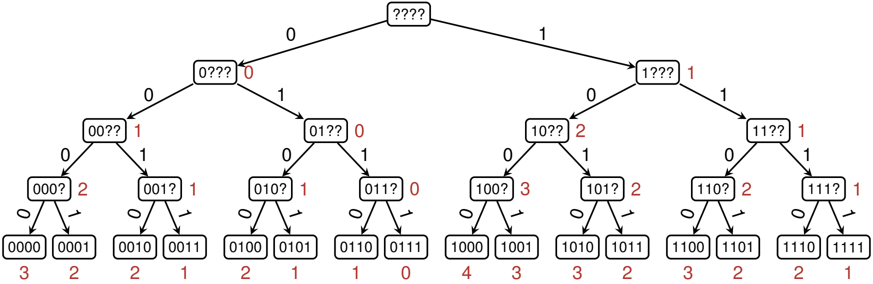

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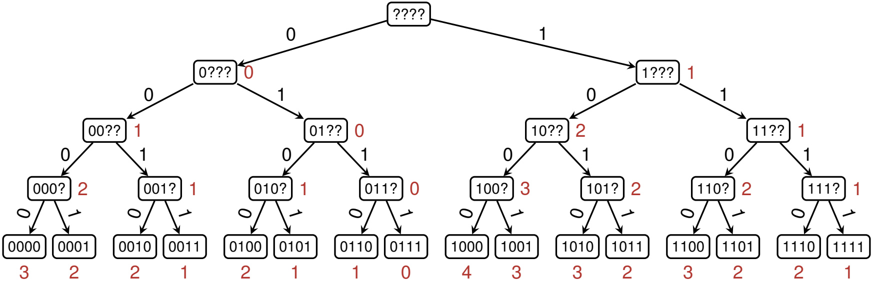

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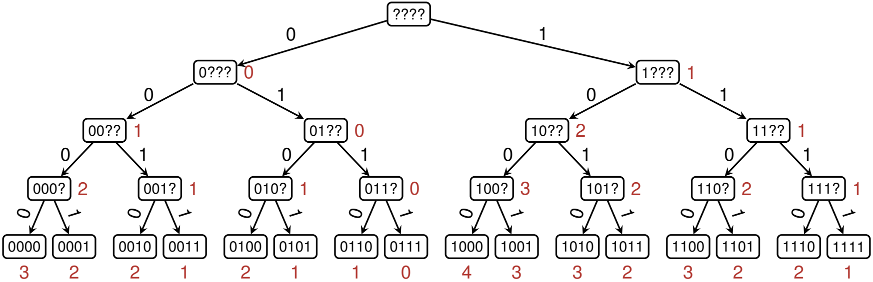

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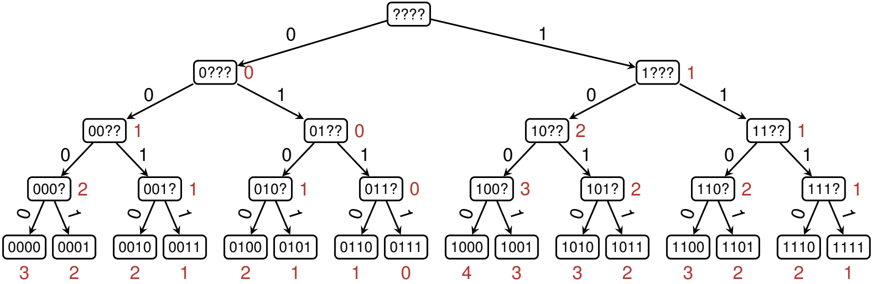

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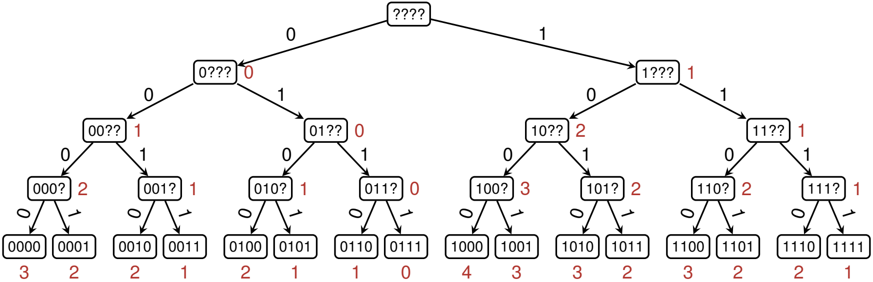

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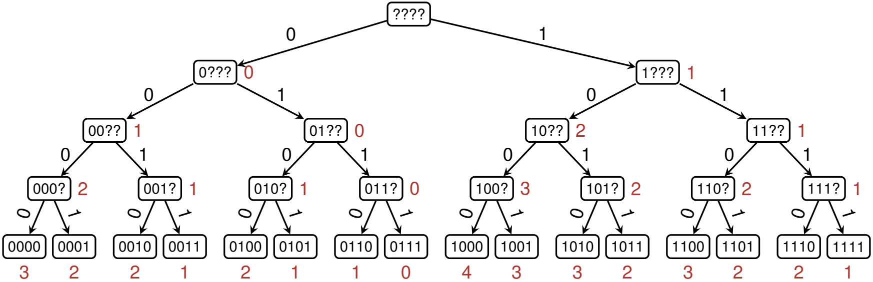

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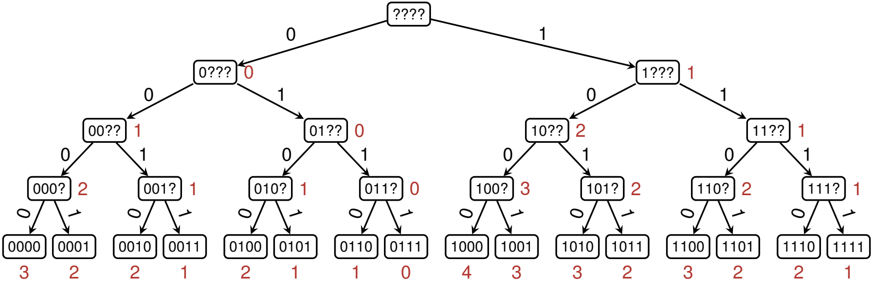

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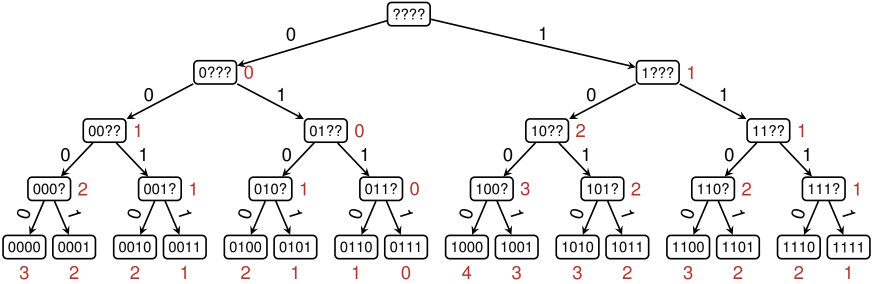

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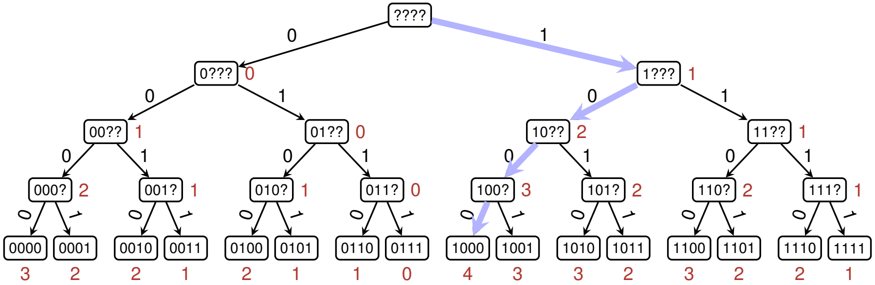

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Much better than the deterministic algorithm!

Analysis of Randomised Weighted Majority (non-examinable)

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The expected number of mistakes of our algorithm $M^{(T)}$ satisfies

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Analysis of Randomised Weighted Majority (non-examinable)

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The expected number of mistakes of our algorithm $M^{(T)}$ satisfies

$$\mathbf{E}\left[M^{(T)}\right] \leq 1 \cdot (1+\delta) \cdot \min_{i \in [n]} m_i^{(T)} + \frac{\ln n}{\delta}.$$

This was a factor of 2 before!

Proof:

- Define a potential function $\Phi^{(t)} = \sum_{i=1}^{n} w_i^{(t)}$, so that $\Phi^{(1)} = n$.
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Can we do better than that?

A Tight Lower Bound

Corollary

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> - Best expert will be good just by chance!

Outline

Choosing the Learning Rate Dynamically (1/2)

Question: How to adjust learning rate if T is not known in advance?

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Approach 1: "The Doubling Trick"

- Algorithm:
  1. For $m = 1, 2, \ldots$
  2. Run a new instance of algorithm on the 2^m rounds $t = 2^m, \ldots, 2^{m+1} - 1$ with "optimal" learning rate (for an algorithm that runs for 2^m steps)
- Analysis before shows that in phase m, number of additional mistakes compared to best expert (regret) is at most $\alpha \cdot \sqrt{2^m}$

$\Rightarrow$ total regret after T steps is at most

$$\frac{\sqrt{2}}{\sqrt{2}-1}\alpha\sqrt{T},$$

where $\alpha = 2 \cdot \sqrt{\ln(n)}$.

Choosing the Learning Rate Dynamically (2/2)

Approach 2

- Algorithm:
  1. Run the randomised WMA with learning rate $\delta_t \approx 1/\sqrt{t}$ in round t
- A modification of our analysis proves for any time-interval $[T/2, T]$:

$$\sum_{t=T/2}^{T} \delta_t \cdot \lambda^{(t)} \cdot p^{(t)} \leq \log(\Phi^{(T/2)}) + (1-\delta_T)^{m_i^{[T/2,T]}}$$

$$\Rightarrow \qquad \mathbf{E}\left[M^{[T/2,T]} \right] \leq \frac{m_i^{[T/2,T]} \cdot \log(1-\delta_T)}{\delta_{T/2}} + \frac{\log(\Phi^{(T/2)})}{\delta_{T/2}}$$

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Approach 3: “Self-Confident Algorithm”

- Algorithm:
  1. Run the randomised WMA with learning rate $\delta_t \approx 1/\sqrt{\min_{i \in [n]} m_i^{(t)}}$ (or $1/\sqrt{M^{(t)}}$) in round t

A More General Setting

New Setup

- At each step, we pick one expert i randomly out of n experts
- That expert i and our algorithm incur a cost $m_i^{(t)}$, but we also observe the costs of all experts (a vector $(m_j^{(t)})_{i=1}^n$)
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Idea of the "Multiplicative Weights-Algorithm"

- In the first iteration, simply pick an expert uniformly at random
- Every expert will be penalised or rewarded through a multiplicative weight-update

The Multiplicative Weights Algorithm

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Initialization: Fix $\delta \leq 1/2$. For every $i \in [n]$, let $w_i^{(1)} := 1$

Update: For $t = 1, 2, \ldots, T$:

- Choose expert i with prop. proportional to $w_i^{(t)}$.
- Observe the costs of all n experts in round t, $m^{(t)}$
- For every expert i, update its weight by:

$$w_i^{(t+1)} = (1 - \delta m_i^{(t)}) w_i^{(t)}$$

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For any expert i, the expected cost of this algorithm is at most

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Derivation is very similar to the ones shown before.

Conclusions

Summary

- Weighted Majority Algorithm
  - natural, simple (and deterministic) algorithm
  - good performance, but could be a factor of 2 worse than the best expert
- Randomised Weighted Majority Algorithm
  - Randomised extension
  - almost optimal performance thanks to randomisation which guards against tailored worst-case instances (cmp. Quick-Sort!)
  - impact of the learning rate: small learning rate gives very good performance guarantees. However, actual performance may depend on the specific data set at hand (cf. simulations!)
- Multiplicative Weight-Update Algorithm
  - further generalisation of the (randomised) weighted majority algorithm

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Outlook

- These algorithms are examples of the Ensemble-Method: Framework combining weak predictions into a strong learner
- A closely related algorithmic approach: Follow the Perturbed Leader
- Similar update schemes are Perceptron and AdaBoost

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