Randomised Algorithms

Lecture 13: Streaming Algorithms

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Lent 2022

Outline

Introduction

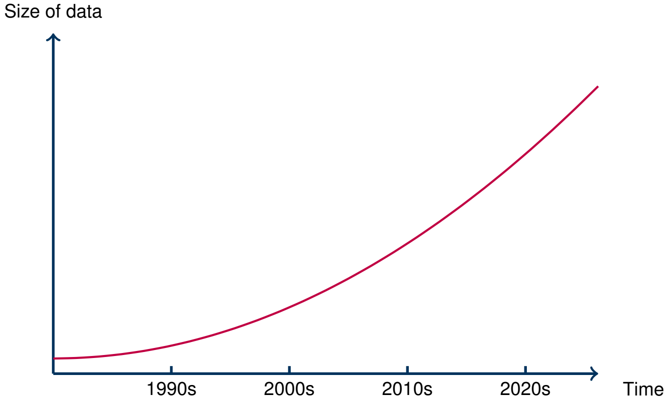
Approximate Counting

Distinct Elements and Frequency Moments

Extra Material (non-examinable): An Algorithm for F_0 in the Turnstile Model

Background of Streaming Algorithms

- The amount of data has been increased exponentially over the last years



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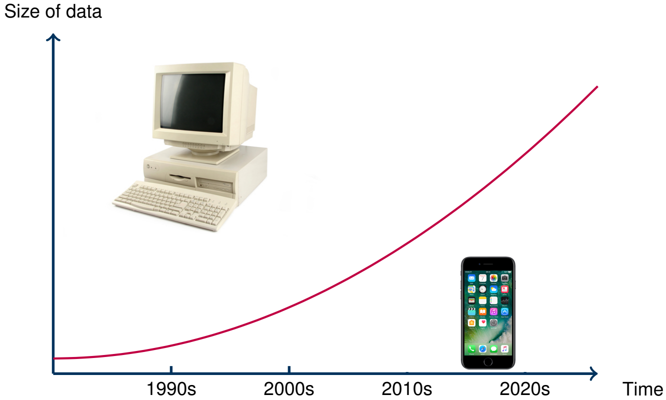
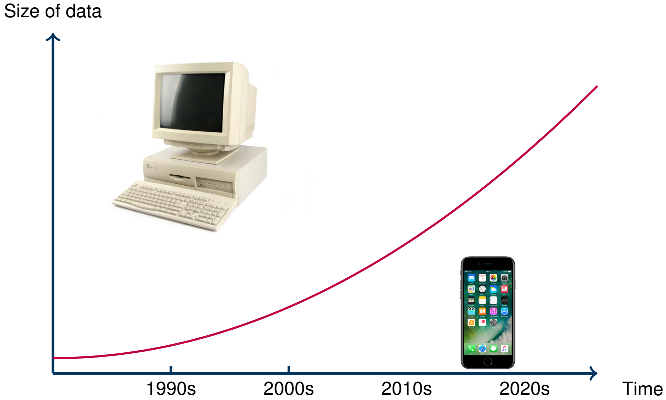


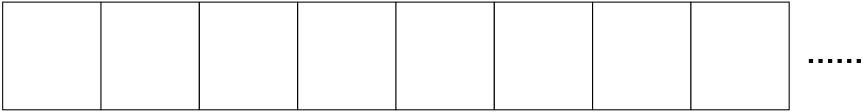
Background of Streaming Algorithms

- The amount of data has been increased exponentially over the last years
- For many applications computational devices' memories are limited
- We need to find good (approximate) solutions without storing the entire input!



Motivation: Analysing Search Engine Queries

Google
PageRank



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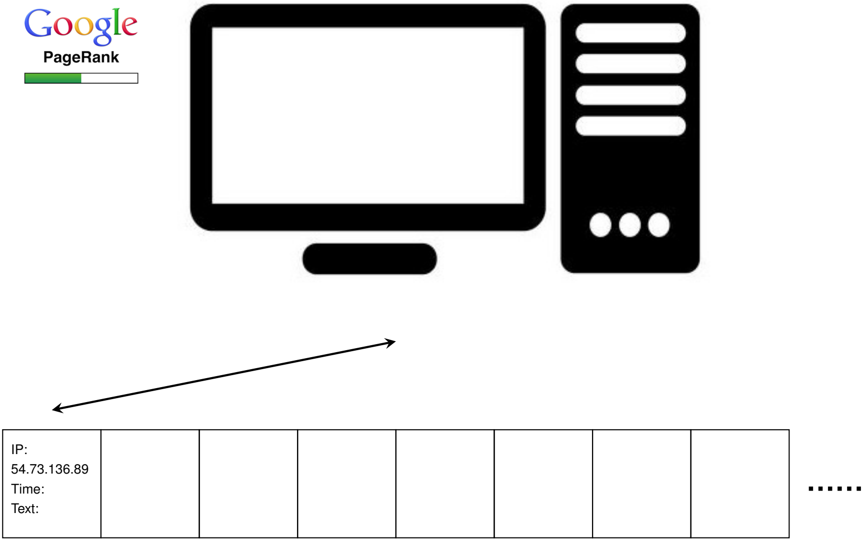

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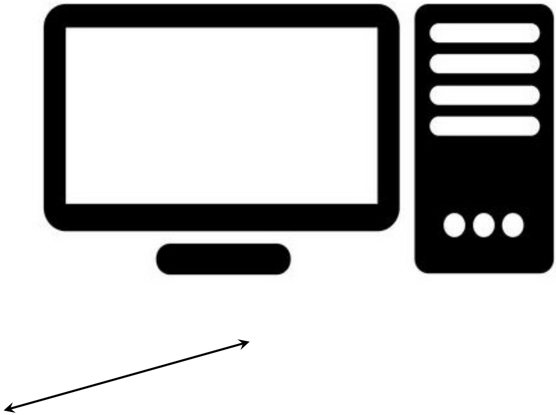

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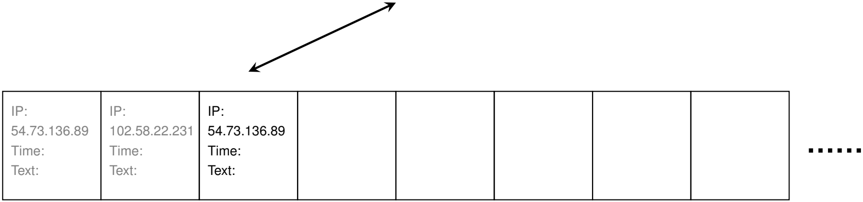

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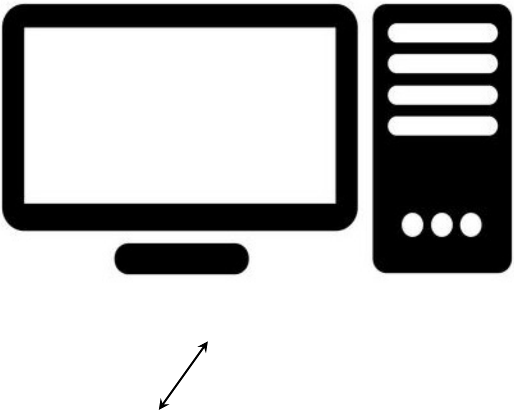

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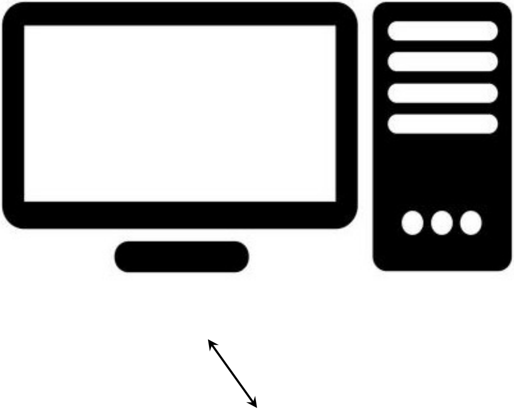

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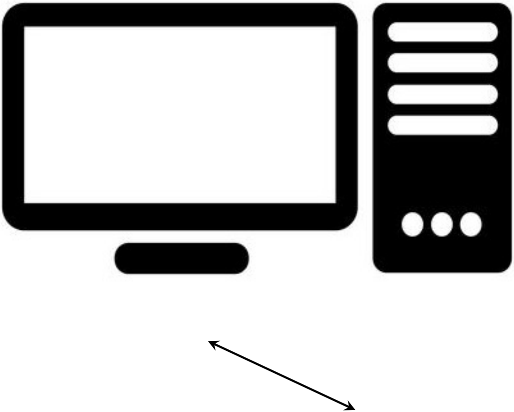

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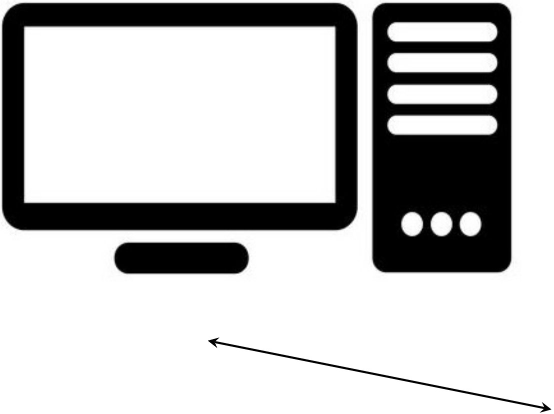

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⇒ We can only read each data item once and in sequential order

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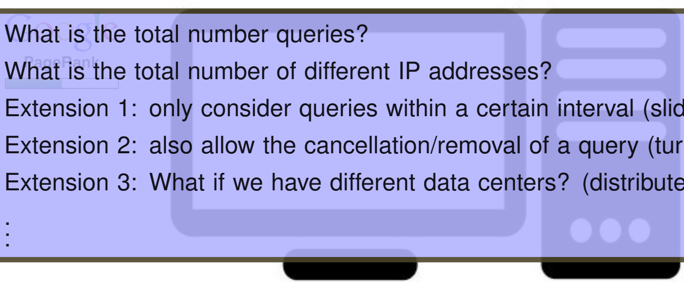

- What is the total number queries?
- What is the total number of different IP addresses?
- Extension 1: only consider queries within a certain interval (sliding window)
- Extension 2: also allow the cancellation/removal of a query (turnstile model)
- Extension 3: What if we have different data centers? (distributed streaming)

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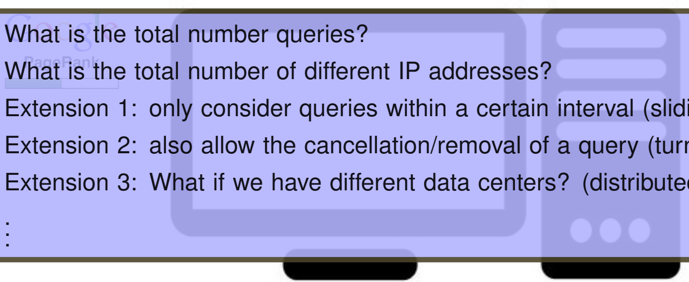

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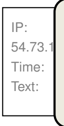

Other Applications:

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- Analysing Buying Histories of Users

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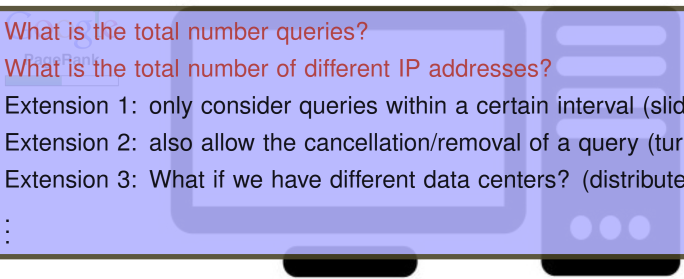

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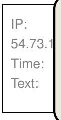

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(ε, δ)-approximation

For confidence parameter δ and approximation parameter ϵ, the algorithm's output Output and the exact answer Exact satisfies

$$\mathbf{P}\left[\,\text{Output} \in (1-\varepsilon, 1+\varepsilon) \cdot \text{Exact}\,\right] \geq 1 - \delta.$$

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Approximate Counting and Morris Algorithm

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An approximate counting algorithm must monitor a sequence of events. At any given time, the algorithm must output an **estimate** of the number of events.

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Intuition: X will be an approximation of $\log_2 n$ (that is, we try to approximate the number of bits of n in binary)

Analysis (1/3)

Lemma (Expectation Analysis)

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By Induction Hypothesis

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Let X_n denote the value of X after n updates. For every $n \geq 0$,

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Hence $\Theta_n := 2^{X_n} - 1$ is an unbiased estimator of n.

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- Since $\mathbf{V}[Z] = \mathbf{E}[Z^2] - \mathbf{E}[Z]^2$,

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This failure probability (estimate) is at least $\frac{1}{2}$ ☹

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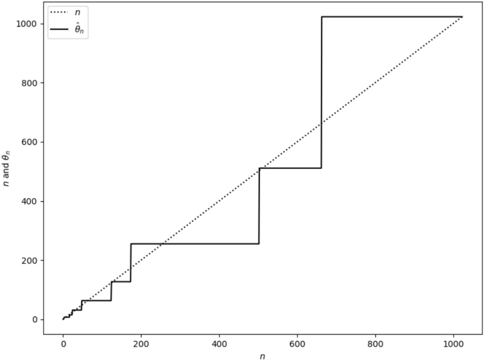
Conclusion

For any $\varepsilon, \delta < 1$, the IMPROVED MORRIS ALG. with $k \geq \frac{1}{2\epsilon^2\delta}$ satisfies:

$$\mathbf{P}\left[\,\left|\overline{\Theta} - n\right| \leq \epsilon \cdot n\,\right] \geq 1 - \delta.$$

Simulation



A run of Morris's algorithm on $n = 1024$ data points

(source: `http://gregorygundersen.com/blog/2019/11/11/morris-algorithm/`)

Outline

Introduction

Approximate Counting

Distinct Elements and Frequency Moments

Extra Material (non-examinable): An Algorithm for F_0 in the Turnstile Model

Norm Estimation: the Alon-Matias-Szegedy algorithm

F_p-norm (Frequency Moments)

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Pairwise Independence

A family of functions $H = \{h \mid h : U \mapsto [n]\}$ is pairwise independent if, for any h chosen uniformly at random from H, the following holds:

1. $h(x)$ is uniformly distributed in $[n] = \{1, 2, \ldots, n\}$ for any $x \in U$;
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Theorem (Fact)

Let n be a prime number, and let $h_{a,b}(x) = (ax + b) \bmod n$. Define

$$H = \{h_{a,b} \mid 0 \leq a, b \leq n - 1\}.$$

Then H is a family of pairwise independent hash functions.

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Since n is not a power of 2, this probability is in fact equal to $\frac{\lfloor n/2^r \rfloor}{n} \approx 1/2^r - o(1)$.

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Recall (ε, δ)-approximation:
$\mathbf{P}\left[\, \text{Output} \in (1-\varepsilon, 1+\varepsilon) \cdot \text{Exact} \,\right] \geq 1 - \delta$

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$$\mathcal{S} = (25, 76, 14, 51, 25, 14, 76, 76, 3, 51, 96, 14, 67, 3, 15, 25, 2, 76, 14, 71)$$

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x	$h(x)$	Binary Representation							$\rho(h(x))$
2	72	1	0	0	1	0	0	0	3
3	100	1	1	0	0	1	0	0	2
14	4	0	0	0	0	1	0	0	2
15	32	0	1	0	0	0	0	0	5
25	9	0	0	0	1	0	0	1	0
51	30	0	0	1	1	1	1	0	1
67	74	1	0	0	1	0	1	0	1
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returned estimate:
$2^{5+1/2} \approx 45.25$

Analysis (1/2)

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$$\mathbf{P}\left[2^{Z+1/2} \geq 3F_0\right] = \mathbf{P}[\,Z \geq p\,] = \mathbf{P}[\,Y_p > 0\,] \leq \frac{F_0}{2^p} \leq \frac{\sqrt{2}}{3}.$$

Let q be the largest integer such that $2^{q+1/2} \leq F_0/3$:

Union Bound: Error $\leq 2 \cdot \frac{\sqrt{2}}{3} < 1$

$$\mathbf{P}\left[2^{Z+1/2} \leq F_0/3\right] = \mathbf{P}[\,Z \leq q\,] \leq \mathbf{P}[\,Y_{q+1} = 0\,] \leq \frac{2^{q+1}}{F_0} \leq \frac{\sqrt{2}}{3}. \qquad \square$$

Final Remarks

- Durand and Flajolet (2003) proposed the LOGLOG algorithm for estimating F_0
- Their algorithm condenses the whole of Shakespeare's works to a table of 256 "small bytes" of 4 bits each
- The estimate of the number of distinct words is $\widetilde{F_0} = 30897$, while the true answer is $F_0 = 28239$, which represents a relative error $+9.4\%$.

```
ghfffghfghgghggggghghheehfhfhhgghghghhfgffffhhhiigfhhffgfiihfhhh
igigighfgihfffghigihghigfhhgeegeghgghhhgghhfhidiigihighihehhhfgg
hfgighigffghdieghhhggghhfghhfiiheffghghihifgggffihgihfggighgiiif
fjgfgjhhjiifhjgehgghfhhfhjhiggghghihigghhihihgiighgfhlgjfgjjjmfl
```

Outline

Introduction

Approximate Counting

Distinct Elements and Frequency Moments

Extra Material (non-examinable): An Algorithm for F_0 in the Turnstile Model

The AMS algorithm (cash register model)

Common approach for designing algorithms in the cash register model:

1. Sample the data items based on hashed values;

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Sampling techniques are usually non-applicable in the turnstile model.

Algorithm to approximate F_2 in the turnstile model

Algorithm to approximate F_2 (simplified description)

```
Choose a **4-wise independent** hash function $h : [n] \to \{-1, 1\}$
$y = 0$
**while** item $(x, \pm)$ from stream $\mathcal{S}$ arrives
    **if** $x$ is inserted **then** $y \leftarrow y + h(x)$
                               **else** $y \leftarrow y - h(x)$
**return** $Z := y^2$
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The algorithm runs in the turnstile model!

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Key Lemma

It holds that $\mathbf{E}[Z] = F_2$ and $\mathbf{V}[Z] \leq 2 \cdot \left(\sum_{i \in \mathcal{S}} m_i^2\right)^2 = 2F_2^2$.

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Key Lemma

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Hence, we can (ε, δ)-approximate F_2, by **running multiple copies of the algorithm in parallel and return the average value.**

Algorithm to approximate F_2 in the turnstile model

Algorithm to approximate F_2 (details)

```
$t = \lceil 6/\varepsilon^2 \rceil$
Choose $t$ 4-wise independent hash function $h_1, \ldots, h_t$, where
    $$h_i : [n] \to \{-1, 1\}$$
$y_i = 0$ for each $i = 1, \ldots, t$
while item $(x, \pm)$ from stream $\mathcal{S}$ arrives
    if $x$ is inserted then $y_i = y_i + h_i(x)$ for every $1 \leq i \leq t$
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return $\frac{1}{t} \cdot \sum_{i=1}^{t} Z_i$, where $Z_i = y_i^2$
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Analysis

With constant probability, the returned value of the algorithm lies in $(1 - \varepsilon, 1 + \varepsilon) \cdot F_2$. Moreover, the space complexity is $O\left((1/\varepsilon^2) \log n\right)$ bits.