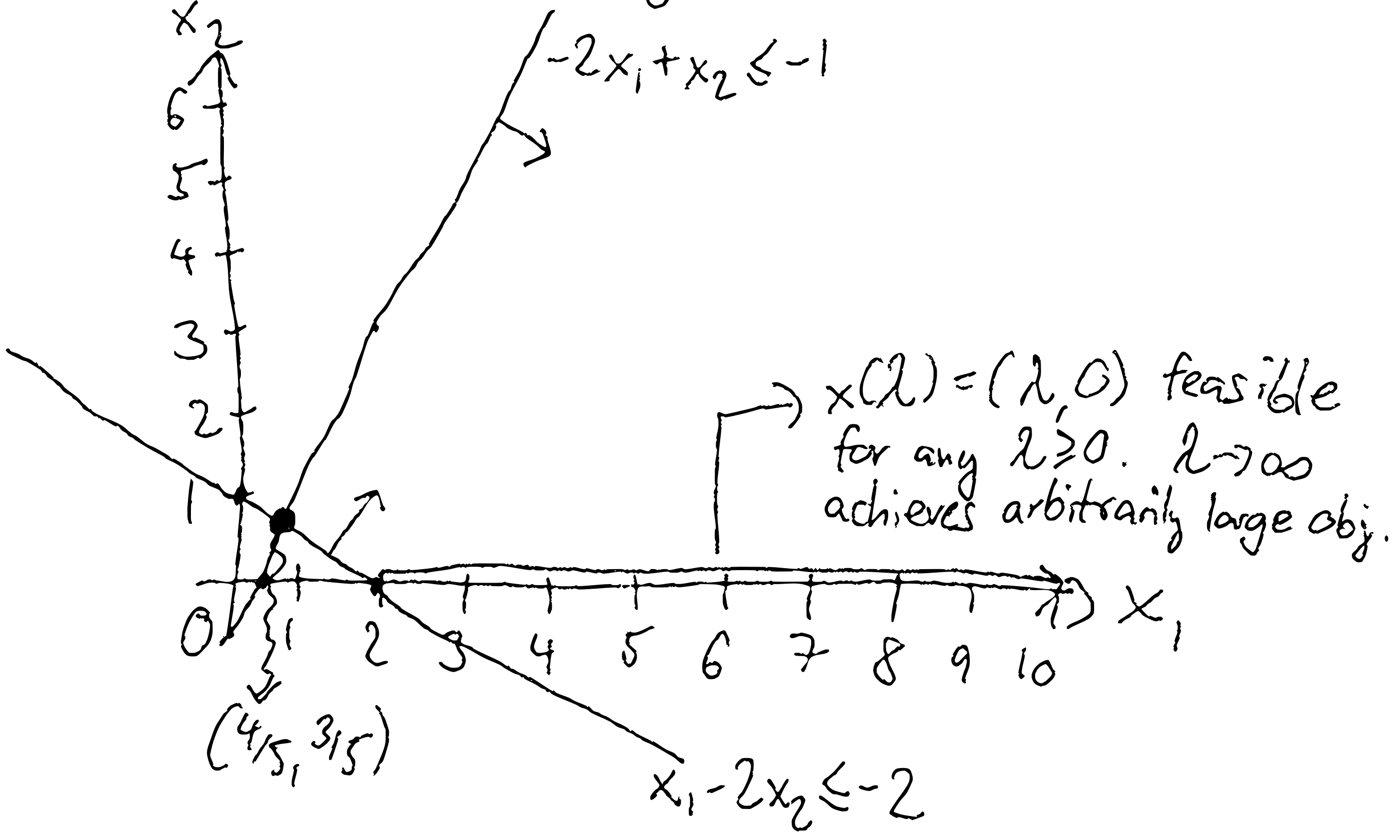
Example Class 4 (LPs, Approx., Spectral)

Question 3

LP:

$$
\begin{array}{lrcrl}
\text{maximise} & x_1 & - & x_2 & \\
\text{subject to} & & & & \\
& -2x_1 & + & x_2 & \leq -1 \\
& -x_1 & - & 2x_2 & \leq -2 \\
& & & x_1, x_2 & \geq 0
\end{array}
$$

Geometry of feasible region:



Let's run one iteration of SIMPLEX

It is already in <u>Standard Form</u>.
Conversion into <u>Slack Form</u>:

$$
\begin{aligned}
z &= x_1 - x_2 \\
x_3 &= -1 + 2x_1 - x_2 \\
x_4 &= -2 + x_1 + 2x_2
\end{aligned}
$$

Can we run SIMPLEX? No, the solution $(-1,-2)$ is <u>not</u> feasible!

$$
\boxed{
\begin{aligned}
z &= -x_0 \\
x_3 &= -1 + 2x_1 - x_2 + x_0 \\
x_4 &= -2 + x_1 + 2x_2 + x_0
\end{aligned}
}
$$

Special Pivot with x_0 entering and x_4 leaving

$\Rightarrow \quad x_0 = 2 - x_1 - 2x_2 + x_4$

$$
\boxed{
\begin{aligned}
z &= -2 + x_1 + 2x_2 - x_4 \\
x_3 &= 1 + x_1 - 3x_2 + x_4 \\
x_0 &= 2 - x_1 - 2x_2 + x_4
\end{aligned}
}
$$

$$\boxed{\begin{aligned} z &= -2 + x_1 + 2x_2 - x_4 \\ x_3 &= 1 + x_1 - 3x_2 + x_4 \\ x_0 &= 2 - x_1 - 2x_2 + x_4 \end{aligned}}$$

$$x_2 = \frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}x_3 + \frac{1}{3}x_4$$

$$\boxed{\begin{aligned} z &= -\frac{4}{3} + \frac{5}{3}x_1 - \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_2 &= \frac{1}{3} + \frac{1}{3}x_1 - \frac{1}{3}x_3 + \frac{1}{3}x_4 \\ x_0 &= \frac{4}{3} - \frac{5}{3}x_1 + \frac{2}{3}x_3 + \frac{1}{3}x_4 \end{aligned}}$$

$$x_1 = \frac{4}{5} - \frac{3}{5}x_0 + \frac{2}{5}x_3 + \frac{1}{5}x_4$$

$$\boxed{\begin{aligned} z &= 0 - x_0 \\ x_2 &= \frac{3}{5} - \frac{1}{5}x_0 - \frac{1}{5}x_3 + \frac{2}{5}x_4 \\ x_1 &= \frac{4}{5} - \frac{3}{5}x_0 + \frac{2}{5}x_3 + \frac{1}{5}x_4 \end{aligned}}$$

feasible solution found!

$$\boxed{\begin{aligned} z &= 0 - x_0 \\ x_2 &= \frac{3}{5} - \frac{1}{5}x_0 - \frac{1}{5}x_3 + \frac{2}{5}x_4 \\ x_1 &= \frac{4}{5} - \frac{3}{5}x_0 + \frac{2}{5}x_3 + \frac{1}{5}x_4 \end{aligned}}$$

Set $x_0 = 0$, restore objective function and express using non-basic variables:

$z = x_1 - x_2$

$$\boxed{\begin{aligned} z &= \frac{1}{5} + \frac{1}{5}x_3 - \frac{1}{5}x_4 \\ x_2 &= \frac{3}{5} - \frac{1}{5}x_3 + \frac{2}{5}x_4 \\ x_1 &= \frac{4}{5} + \frac{2}{5}x_3 + \frac{1}{5}x_4 \end{aligned}}$$

x_3 enterring, x_2 leaving variable:

$$x_3 = 3 - 5x_2 + 2x_4$$

$$\boxed{\begin{aligned} z &= \frac{4}{5} - x_2 + \frac{1}{5}x_4 \\ x_3 &= 3 - 5x_2 \underline{+ 2x_4} \\ x_1 &= 2 - 2x_2 \underline{+ x_4} \end{aligned}}$$

x_4 can be increased arbitrarily, and SIMPLEX returns <u>unbounded</u>!

$\boxed{Q4}$ Shortest Paths via LPs

Part a) What happens if there is a negative weight cycle?

Let $c = (v_0, v_1, \dots, v_\ell = v_0)$ with

$$\sum_{i=1}^{\ell} w(v_{i-1}, v_i) < 0$$

We have ℓ linear constraints:

$$d_{v_i} \leq d_{v_{i-1}} + w(v_{i-1}, v_i) \quad 1 \leq i \leq \ell$$

Combining them yields:

$$\begin{aligned} d_{v_\ell} &\leq d_{v_{\ell-1}} + w(v_{\ell-1}, v_\ell) \\ &\leq d_{v_{\ell-2}} + w(v_{\ell-2}, v_{\ell-1}) + w(v_{\ell-1}, v_\ell) \\ &\;\;\vdots \\ &\leq \underset{\substack{\parallel \\ d_{v_\ell}}}{d_{v_0}} + \underbrace{\sum_{i=1}^{\ell} w(v_{i-1}, v_i)}_{<0} \quad \lightning \end{aligned}$$

$\Rightarrow$ LP is not feasible (even if c is not reachable!)

Q4

Part b) If there is no negative weight cycle, there exists d_v, $v \in V$ satisfying all triangle inequalities:

$$d_v \leq d_u + w(u,v) \quad \forall \{u,v\} \in E(G)$$

Thus the LP is feasible, as the actual distances d_v form a solution.

Now let $\bar{d}$ be the optimal solution of the LP. We need to show:

$$\bar{d}_t \leq d_t \qquad (\geq \text{ was shown})$$

Let $P = (\underset{\overset{\shortparallel}{s}}{v_0}, v_1, \dots, \underset{\overset{\shortparallel}{t}}{v_\ell})$ be a shortest path.

Then: $$\bar{d}_{v_\ell} \leq \bar{d}_{v_{\ell-1}} + w(v_{\ell-1}, v_\ell)$$
$$\leq \bar{d}_{v_{\ell-2}} + w(v_{\ell-2}, v_{\ell-1}) + w(v_{\ell-1}, v_\ell)$$
$$\vdots$$
$$\leq \underbrace{\bar{d}_{v_0}}_{=0} + \underbrace{\sum_{i=1}^{\ell} w(v_{i-1}, v_i)}_{= d_t} = d_{v_\ell}$$

$\boxed{Q4}$

Part c) To solve the SSSP problem, change objective function to:

$$\text{maximise} \sum_{v \in V} d_v$$

L9/10

$\boxed{\text{Question 2}}$ Derandomisation for MAX-CUT

Randomised Algorithm:

- For each $u \in V$, let $X_u \sim Ber(1/2)$
- Let $S := \{v \in V : X_v = 1\}$ and $Z := e(S, S^c) = |E(S, S^c)|$

Derandomisation:

- Take any order of vertices $u_1, u_2, \dots, u_n$
- Decompose:

$$E[Z] = \frac{1}{2} \cdot E[Z \mid X_{u_1} = 1] + \frac{1}{2} \cdot E[Z \mid X_{u_1} = 0]$$

Pick value $z_1 \in \{0,1\}$ "greedily" and continue:

$$E[Z \mid X_{u_1} = z_1, \dots, X_{u_i} = z_i]$$
$$= \frac{1}{2} E[Z \mid X_{u_1} = z_1, \dots, X_{u_i} = z_i, X_{u_{i+1}} = 1]$$
$$+ \frac{1}{2} E[Z \mid X_{u_1} = z_1, \dots, X_{u_i} = z_i, X_{u_{i+1}} = 0]$$

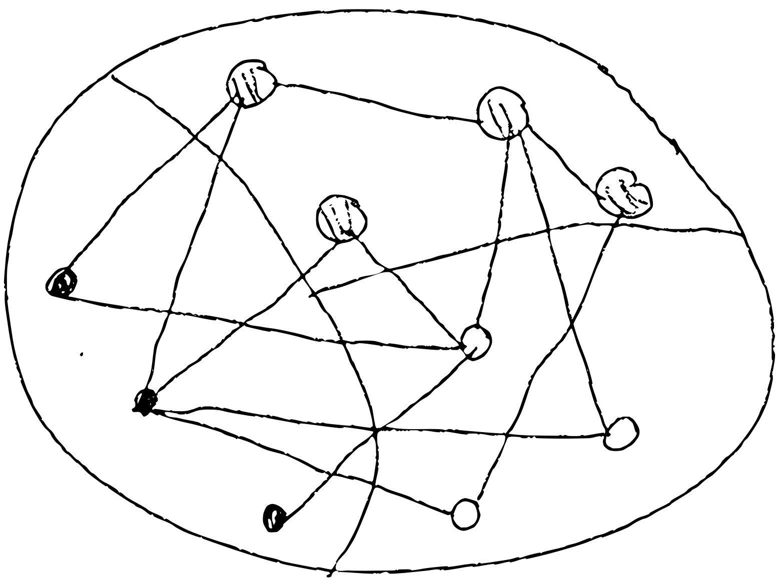
$\Rightarrow$ ensures we have a 2-approximation (deterministically)

$\boxed{\text{Question 2}}$ (continuation)

What does

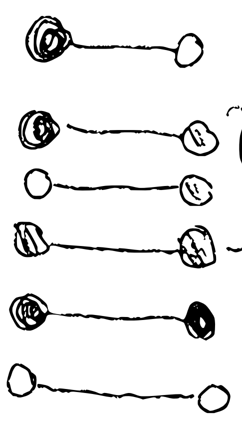
$$E[Z \mid X_{u_1} = z_1, \dots, X_{u_i} = z_i] \text{ mean?}$$

(and can we compute it in poly-time)?



3 parts in the graph:

- Black vertices: $X_u = 1$
- White vertices: $X_u = 0$
- Gray vertices: $X_u = ?$



→ contributes 1 to Z

→ contribute 1/2 to Z

→ contribute 0 to Z

Using Linearity of Expectations, we can compute $E[Z \mid \dots]$ in polynomial time.

Interpretation: Algorithm picks next vertex u_i and colors it with the minority color of neighbours!

$\boxed{\text{Question 5}}$ Set-Cover LP

$$\text{minmise} \sum_{S \in \mathcal{F}} c(S)\, y(S)$$

$$\text{s.t.} \quad \sum_{S \in \mathcal{F}: x \in S} y(S) \geq 1$$

$$y(S) \in [0, 1]$$

additionally we know that each $x \in X$ appears in at most K subsets $S \in \mathcal{F}$.

$$1 \leq \underbrace{\sum_{S \in \mathcal{F}: x \in S}}_{K \text{ addends}} y(S) \leq K \cdot \max_{S \in \mathcal{F}: x \in S} y(S)$$

$$\Rightarrow \max_{S \in \mathcal{F}: x \in S} y(S) \geq \frac{1}{K} \quad \text{for any LP sol.}$$

Thus deterministic rounding will include for each $x \in X$ at least one set $S \in \mathcal{F}$ with $x \in S$. Similar to the 2-approximation ratio of VERTEX-COVER, this gives a K-approximation. ∎

Question 6

Let $f \cdot P = -f \cdot P$.

Define $w = \underset{u}{\operatorname{argmax}} \frac{|f(u)|}{d(u)} > 0$

$\uparrow$ since $f \neq \vec{0}$

$fP = -f \cdot P \Rightarrow$

$$f(w) = -\sum_{v: v \sim w} \frac{f(v)}{d(v)} \qquad (*)$$

$$\Rightarrow |f(w)| \leq \sum_{v: v \sim w} \frac{|-f(v)|}{d(v)}$$

$$\leq \sum_{v: v \sim w} \frac{|f(w)|}{d(w)} = |f(w)|$$

$$\Rightarrow \forall v \sim w: f(v) \in \left\{ \frac{-|f(w)|}{d(w)} \cdot d(v), \frac{|f(w)|}{d(w)} \cdot d(v) \right\}$$

By $(*) \Rightarrow f(v) = -\frac{f(w)}{d(w)} \cdot d(v) \quad \forall v \sim w$

Iterating this, and since G is connected, f assigns strictly positive and negative values to adjacent vertices $\Rightarrow$ G is 2-colourable $\Rightarrow$ G is bipartite

L4/5 Q2

$\Rightarrow$ Simple Random Walk is <u>periodic</u>

$\blacksquare$