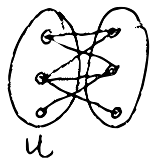
EXAMPLE CLASS 2 RANDOMISED ALG.

Question 2

SRW transition matrix

<u>Part 1:</u> G is bipartite $\Rightarrow$ P is periodic

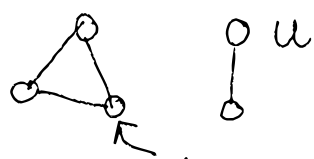


$$p^t_{u,u} \begin{cases} =0 & \text{if } t \text{ is odd} \\ >0 & \text{if } t \text{ is even} \end{cases}$$

$$\gcd\{t \geq 1 : p^t_{u,u} > 0\} = 2 \neq 1$$

<u>Part 2:</u> G is not bipartite $\overset{?}{\Rightarrow}$ P is aperiodic

G:



G is <u>not</u> bipartite, but

$$\gcd\{t \geq 1 : p^t_{u,u} > 0\} = 2 \neq 1,$$

so P may still be periodic. $\lightning$

How to fix? Add that G is <u>connected</u>

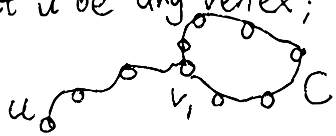
Question 2

Part 3: G is not bipartite & connected $\Rightarrow$ P aperiodic

G is not bipartite $\Rightarrow \exists$ odd cycle

$$C = (v_1, v_2, \dots, v_\ell = v_1), \quad \ell \text{ odd}$$

Let u be any vertex; this vertex is connected to C:



$$P_{u,u}^{\overbrace{2\cdot d(u,v_1)+\ell}^{\text{odd}}} > 0, \quad \text{and} \quad P_{u,u}^{t} > 0 \text{ for any even } t$$

$\Rightarrow \exists\, t_0 \in \mathbb{N}$ such that $P_{u,u}^{t_0} > 0$ and $P_{u,u}^{t_0+1} > 0$

$\Rightarrow \gcd\{t \geq 1 : P_{u,u}^{t} > 0\} = 1$ for all $u \in V$

$\Rightarrow P$ is aperiodic.

Question 4

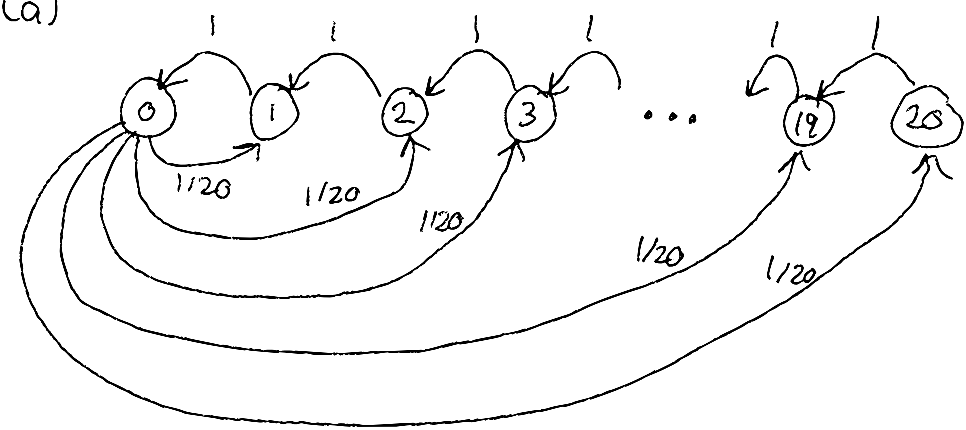
X_n = state of the bus stop after minute n

$\Omega = \{0, 1, 2, \dots, 20\}$

$\downarrow$

At the current minute the bus is at the stop

(a)



(b) π = stationary distribution $\Rightarrow$ waiting i minutes $= \pi_i$

$\pi \cdot P = \pi$ and $\sum_{i=1}^{20} \pi_i = 1$

$$\left.\begin{aligned} \pi_1 \cdot 1 &= \pi_0 \\ \pi_2 \cdot 1 + \pi_0 \cdot \frac{1}{20} &= \pi_1 \end{aligned}\right\} \Rightarrow \pi_2 = \frac{19}{20}\pi_0$$

$$\pi_3 \cdot 1 + \pi_0 \frac{1}{20} = \pi_2 \quad \Big\} \Rightarrow \pi_3 = \frac{18}{20}\pi_0$$

$\vdots$

$$\pi_0 \cdot \frac{1}{20} = \pi_{20}$$

$$\forall k \geq 1: \pi_k = \frac{21-k}{20}\pi_0 \Rightarrow \pi_0 = \frac{2}{23}$$

Question 4

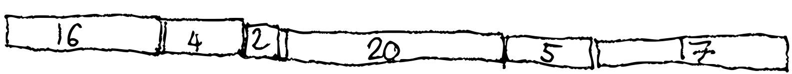
(c) expected waiting time?

$$E_{\pi}[T] = \sum_{k=0}^{20} k \cdot \pi_k$$

$$= 0 \cdot \pi_0 + \sum_{k=1}^{20} k \cdot \frac{21-k}{20} \cdot \frac{2}{23} \approx 6.7$$

Why is this number not:

$$\underbrace{\frac{1}{2} \cdot (1+20)}_{\text{average length of an interval between two arrivals}} \approx 10.5 \Rightarrow \frac{1}{2} \cdot (1+10.5) = 5.75 \ ?$$



We are sampling a "random" point in time, but not a "random" interval between two buses $\Rightarrow$ more likely to sample a point from a longer interval!

(a.k.a. Feller's paradox in Palm Calculus!)

Question 6

$$\|\mu \cdot P - \nu P\|_{tv} \overset{!}{\leqslant} \|\mu - \nu\|_{tv}$$

<u>Note:</u> We don't need any "nice" properties of P like irreducible or aperiodic!

$$\|\mu P - \nu P\|_{tv} = \frac{1}{2} \sum_{x\in\Omega} \Big| \sum_{y\in\Omega} \mu(y) P(y,x) - \sum_{y\in\Omega} \nu(y) P(y,x) \Big|$$

$$= \frac{1}{2} \sum_{x\in\Omega} \Big| \sum_{y\in\Omega} (\mu(y) - \nu(y)) P(y,x) \Big|$$

$$\leqslant \frac{1}{2} \cdot \sum_{x\in\Omega} \sum_{y\in\Omega} |\mu(y) - \nu(y)| \cdot \underbrace{P(y,x)}_{\geq 0}$$

$$= \frac{1}{2} \sum_{y\in\Omega} |\mu(y) - \nu(y)| \underbrace{\sum_{x\in\Omega} P(y,x)}_{=1}$$

$$= \|\mu - \nu\|_{tv}$$

If $\mu = P^t_{x,\cdot}$ (distribution of chain after t steps, starting from x)
and $\nu = \pi$, then:

$$\|P^{t+1}_{x,\cdot} - \pi\|_{tv} \leqslant \|P^t_{x,\cdot} - \pi\|_{tv}$$

Hence the total var. distance from π is non-increasing in t.
(This even works if chain has <u>different</u> stationary distributions!)

Question 12

- $G = (V, E)$ undirected
- simple random walk

First, let us assume that G is <u>connected</u>

$$\forall \{u,v\} \in E(G): \quad h(u,v) \le 2 \cdot |E|$$

$$h(u,u) = \frac{1}{\pi_u} = \frac{2|E|}{\deg(u)}$$

Recurrence Formula: [Lecture 4, slide 7]

$$h(x,y) = 1 + \sum_{z \in \Omega \setminus \{y\}} P(x,z) \cdot h(z,y)$$

This equation also works for $x = y = u$:

$$\Rightarrow \quad h(v,v) = 1 + \sum_{w: \{v,w\} \in E} P(v,w) \cdot h(w,v)$$

this uses "connected"

$$\ge 1 + P(v,u) \cdot h(u,v)$$

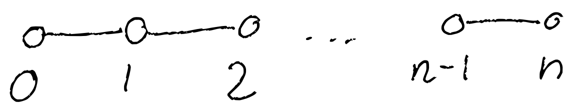
$$\Rightarrow \underline{h(u,v)} \le \frac{h(v,v) - 1}{P(v,u)} \le \frac{h(v,v)}{P(v,u)} = \frac{\frac{2|E|}{\deg(v)}}{\frac{1}{\deg(v)}} = \underline{2|E|}$$

<u>Additional Q:</u> (thanks to student) What happens if G is <u>not connected</u>?

Then restrict random walk to connected component C of edge $\{u,v\} \in E$. Then $h(v,v) = \frac{2|E(C)|}{\deg(v)} \le \frac{2|E|}{\deg(v)}$

Question 14

First Part: G is a path with $V=\{0,1,\dots,n\}$, n even.

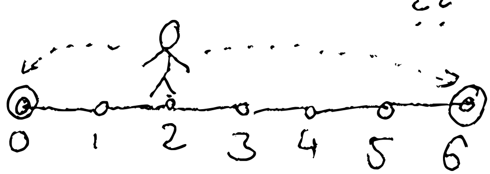


Recall: $h(0,n) = n^2 = h(n,0)$

<u>Q</u>: Which is the worst-case start vertex?

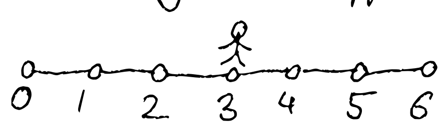
<u>A</u>: It can't be 0 or n, since when the SRW starts from $k \in \{1,2,\dots,n-1\}$ it will hit 0 or n at some point and then the walk still needs $h(0,n)$ (or $h(n,0)$) expected steps.

$$\Rightarrow \quad t_{cov}(k) = \underbrace{h(k,\{0,n\})}_{= \,??} + \underbrace{h(0,n)}_{= \, n^2}$$



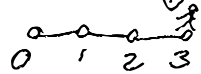
Intuitively, the highest hitting time happens when $k = \frac{n}{2}$

$\Rightarrow$



X_t

equivalent to

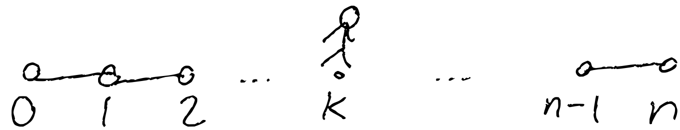


$\min\{X_t, 6-X_t\}$

$$\Rightarrow h\left(\tfrac{n}{2},\{0,n\}\right) = h\left(\tfrac{n}{2},0\right) = \frac{n^2}{4}$$

Question 14 (continuation)

We need to analyse:



where $K \in \{0, 1, \dots, n\}$. This is also known as <u>"Gambler's Ruin Problem"</u>; you start with a capital of K. If you reach 0, you go broke. If you reach n, the opponent goes broke.

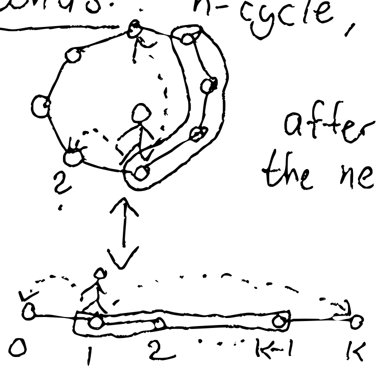
Solving linear equations (difference equations) yields:

$$h(K, \{0, n\}) = K \cdot (n - K)$$

$\Rightarrow$ maximised for $K = \frac{n}{2}$ (as anticipated...)

Hence $t_{cov} = n^2 + \frac{1}{4} n^2 = \underline{\underline{\frac{5}{4} n^2}}$.

<u>Bonus:</u> ? n-cycle, $n \in \mathbb{N}$ (odd or even)



after visiting $K-1$ vertices, the next vertex is visited after $1 \cdot (K-1)$ expected steps

$$\Rightarrow t_{cov} = \sum_{K=2}^{n} 1 \cdot (K-1) = \underline{\underline{\frac{1}{2} n (n-1)}}$$