Quantum Computing: Exercise Sheet 1

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1. Which of the following are possible states of a qubit?
   (a) $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   (b) $\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle$
   (c) $0.7|0\rangle + 0.3|1\rangle$
   (d) $0.8|0\rangle + 0.6|1\rangle$
   (e) $\cos\theta|0\rangle + i\sin\theta|1\rangle$
   (f) $\cos^2\theta|0\rangle - \sin^2\theta|1\rangle$
   (g) $\left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$

   For each valid state among the above, give the probabilities of observing $|0\rangle$ and $|1\rangle$ when the system is measured in the standard computational basis.

   What are the probabilities of the two outcomes when the state is measured in the basis $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$?

2. A two-qubit system is in the following state

$$\frac{1}{\sqrt{30}}(|00\rangle + 2i|01\rangle - 3|10\rangle - 4i|11\rangle)$$

   The first qubit is measured and observed to be 1. What is the state of the system after the measurement? What is the probability that a subsequent measurement of the second qubit will observe a 1?

3. Find the eigenvalues and associated eigenvectors of $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

4. Uniqueness of dimension: Suppose that $|v_1\rangle, \ldots, |v_n\rangle$ is a basis for V. Let $|u_1\rangle, \ldots, |u_{n+1}\rangle$ be any collection of $n+1$ vectors. Show that they cannot all be linearly independent, i.e. one of them must be expressible as a linear combination of the others.

5. Express each of the two linear operators $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ as a linear combination of outer products of computational basis vectors.

6. Show that the Hadamard matrix and the three Pauli matrices are unitary.

7. If I is the 2-dimensional identity matrix and H is the Hadamard operator, give matrix representations of the operators $I \otimes H$ and $H \otimes I$.

8. Let $|\psi\rangle = \alpha 0 + \beta|1\rangle$, if $|\psi\rangle$ is measured using measurement operators $M_0 = |0\rangle\langle 0|$ and $M_1 = |1\rangle\langle 1|$ verify that $p(M_0) = |\alpha|^2$ and $p(M_1) = |\beta|^2$.

9. Show that unitary operations are norm preserving. That is, if U is unitary, then the norm of $U|\psi\rangle$ equals the norm of $|\psi\rangle$, for all $|\psi\rangle$.

10. For the Pauli matrices X, Y and Z, show that $XY = iZ$.

11. Suppose a two-qubit system is in the state $0.8|00\rangle + 0.6|11\rangle$. A Pauli X gate (i.e. a NOT gate) is applied to the second qubit and a measurement performed (on each qubit) in the computational basis. What are the probabilities of the possible measurement outcomes?

12. Show that the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ cannot be expressed as a tensor product of two single qubit states.

    Hint: start with a general expression of a tensor product of two single qubit states, $(\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$ and multiply out.

13. We wish to distinguish two quantum states:

    - In the first case, the state is either $|0\rangle$ or $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$.
    - In the second case, the state is either $|0\rangle$ or $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

    Which of these cases do you expect to be able to distinguish with higher probability. Verify your answer using the Helstrom-Holevo bound.

14. Describe (qualitatively) how, if cloning were possible, then the no-signalling principle could be violated.