Quantum Computing (CST Part II)

Lecture 13: Quantum Error Correction

We have learned that it is possible to fight entanglement with entanglement.
John Preskill

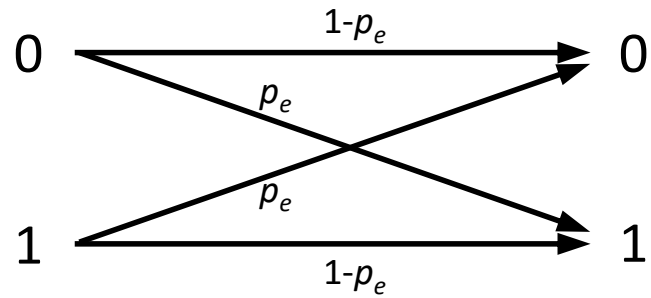
Why do we need quantum error correction?

Modern (classical) digital computers are, to all intents and purposes, error-free. The same cannot, however, be said of quantum computers, therefore error correction is required, in particular:

- Even though near-term hybrid quantum-classical algorithms have been conceptualised to achieve classically-intractable simulations, even in the presence of noise, it is becoming increasingly apparent that some amount of error correction is crucial to achieve satisfactory performance.
- More fundamentally, in order to assert that quantum algorithms will indeed achieve super-classical performance in practise, it is necessary to understand the "asymptotic significance" of quantum errors, and the possibility and efficiency of correction.

Classical errors: the binary symmetric channel

One of the simplest models for single-bit (classical) errors is the *binary symmetric channel*, in which each possible state of the bit, 0 and 1 "flips" to the other with some probability p_e:



Note that, without loss of generality we can assume $p_e \leq 0.5$, because if $p_e > 0.5$ then it is more likely than not that a bit-flip has occurred, so we can interpret a received 0 as a 1 and vice-versa. In the case where $p_e = 0.5$ we cannot recover any information from the channel.

Classical error correction: the three-bit repetition code

If we wish to send a single bit over a binary symmetric channel, then we can encode the bit, by simply repeating it three times. That is, if we wish to transmit a 0, we send three bits (sequentially) in the state 0, and likewise for 1. This can be denoted as:

$$0 \to 000$$
$$1 \to 111$$

Once the three bits have been received, they are decoded by a "majority vote". So in order for an error to occur, it is necessary that either two of the three bits have been flipped (which can occur in three different ways), or all three have been, that is:

$$p'_e = 3p_e^2(1 - p_e) + p_e^3$$

Which is less than p_e if $p_e < 0.5$. Typically, p_e is small, and we can describe this as suppressing the error to $\mathcal{O}(p_e^2)$.

Complicating factors with quantum error correction

Ostensibly, it appears that we cannot directly transfer classical error correction techniques to the problem of quantum error correction for three reasons:

1. The no-cloning principle forbids the copying of quantum states.
2. Measurement destroys quantum information.
3. Quantum states are continuous: $\alpha |0\rangle + \beta |1\rangle$. Therefore quantum errors are also continuous: $\alpha |0\rangle + \beta |1\rangle \rightarrow (\alpha + \epsilon_0) |0\rangle + (\beta + \epsilon_1) |1\rangle$

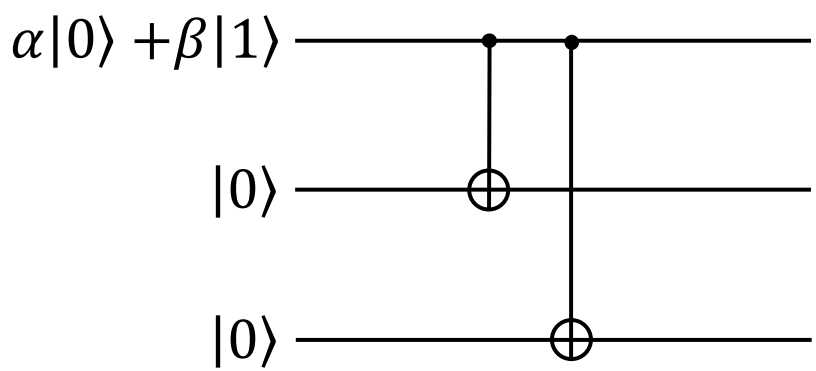
Nevertheless, we shall see that with some ingenuity we *can* correct quantum errors.

The three-qubit bit-flip code

The three-bit repetition code guarantees to return the correct bit value, so long as at most one of the bits in the code is flipped. We now use this as inspiration for the *three-qubit bit-flip code*, in which entanglement rather than cloning plays the role of the repetition. That is, we encode the computational basis states:

$$|0\rangle \to |000\rangle$$
$$|1\rangle \to |111\rangle$$

Which is achieved using the following circuit:

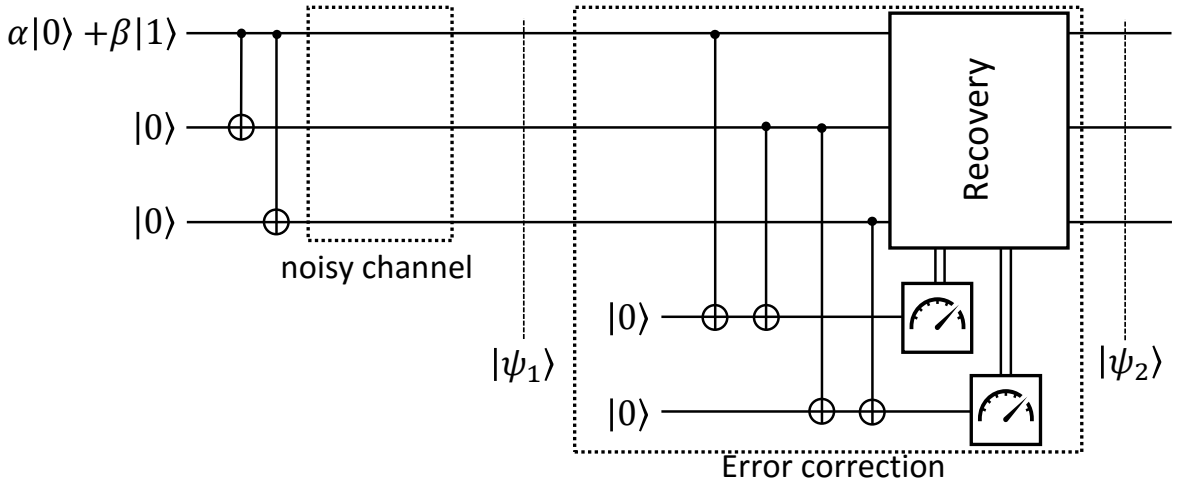


This has the following action on an arbitrary qubit state:

$$(\alpha\,|0\rangle + \beta\,|1\rangle)\,|0\rangle^{\otimes 2} \to \alpha\,|000\rangle + \beta\,|111\rangle$$

The three-qubit bit-flip code: error detection and recovery

To detect and recover errors, we supplement the circuit with two ancillas that we use for error detection:



We can thus detect and recover single-qubit bit-flips:

Bit-flip	$\|\psi_1\rangle$	M_1	M_2	**Recovery**	$\|\psi_2\rangle$
-	$\alpha\|000\rangle + \beta\|111\rangle$	0	0	$I \otimes I \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
1	$\alpha\|100\rangle + \beta\|011\rangle$	1	0	$X \otimes I \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
2	$\alpha\|010\rangle + \beta\|101\rangle$	1	1	$I \otimes X \otimes I$	$\alpha\|000\rangle + \beta\|111\rangle$
3	$\alpha\|001\rangle + \beta\|110\rangle$	0	1	$I \otimes I \otimes X$	$\alpha\|000\rangle + \beta\|111\rangle$

That is, we have made comparative *parity-check* measurements that tell us only about the error and not about the quantum state itself, and so these measurements have not destroyed the quantum state.

The three-qubit phase-flip code

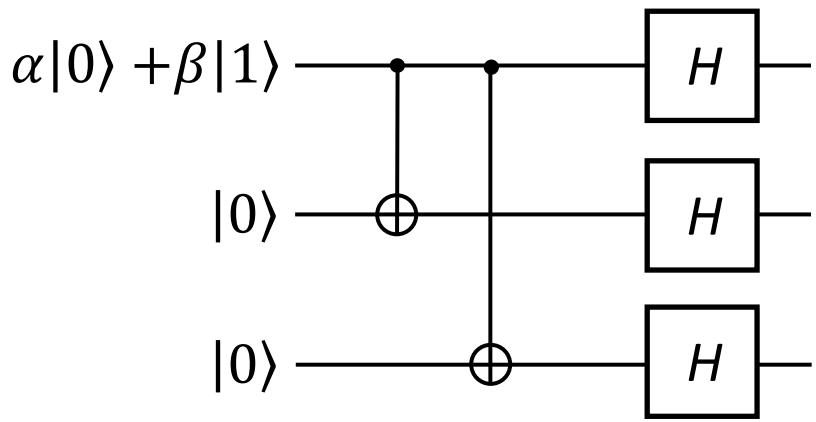
The three-qubit bit-flip code demonstrates how we can overcome two of the possible problems with quantum error correction that we previously identified:

- We can use entanglement to enable repetition.
- We can detect errors using parity-check measurements that do not destroy the quantum information.

However, we still have not addressed the fact that quantum states are continuous. To begin to do this, we'll look at an error correction code for a different type of error. The three-qubit phase-flip code has the following action on an arbitrary single-qubit state:

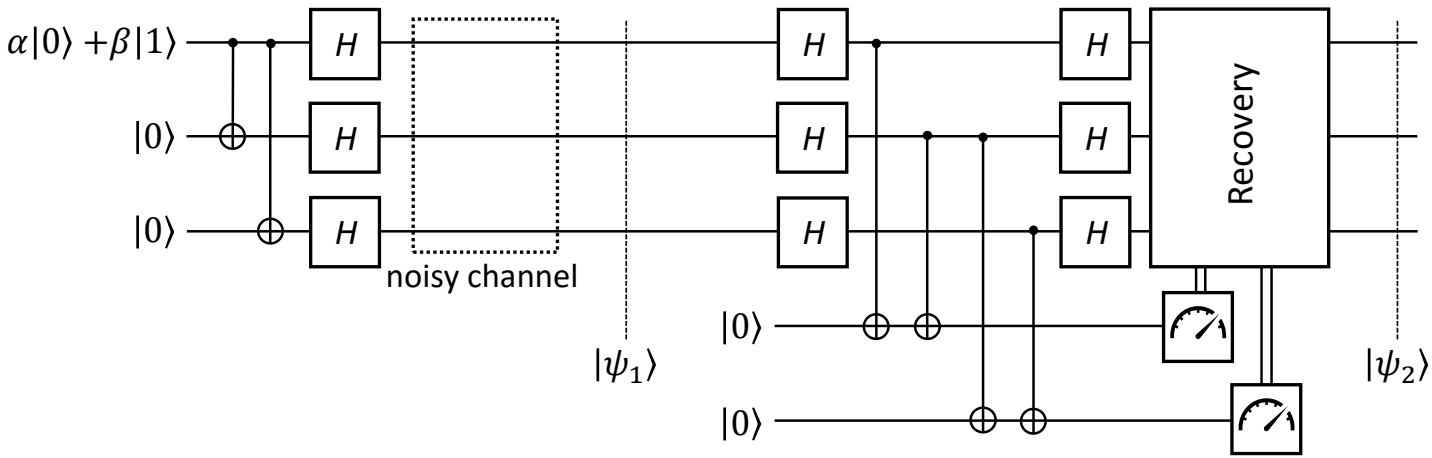
$$(\alpha |0\rangle + \beta |1\rangle) |0\rangle^{\otimes 2} \rightarrow \alpha |+++\rangle + \beta |---\rangle$$

Which is achieved by the following circuit:



Three-qubit phase-flip code: error detection and recovery

Once again, to detect and recover errors, we supplement the circuit with two ancillas that we use for error detection:



By definition, a phase flip sends:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

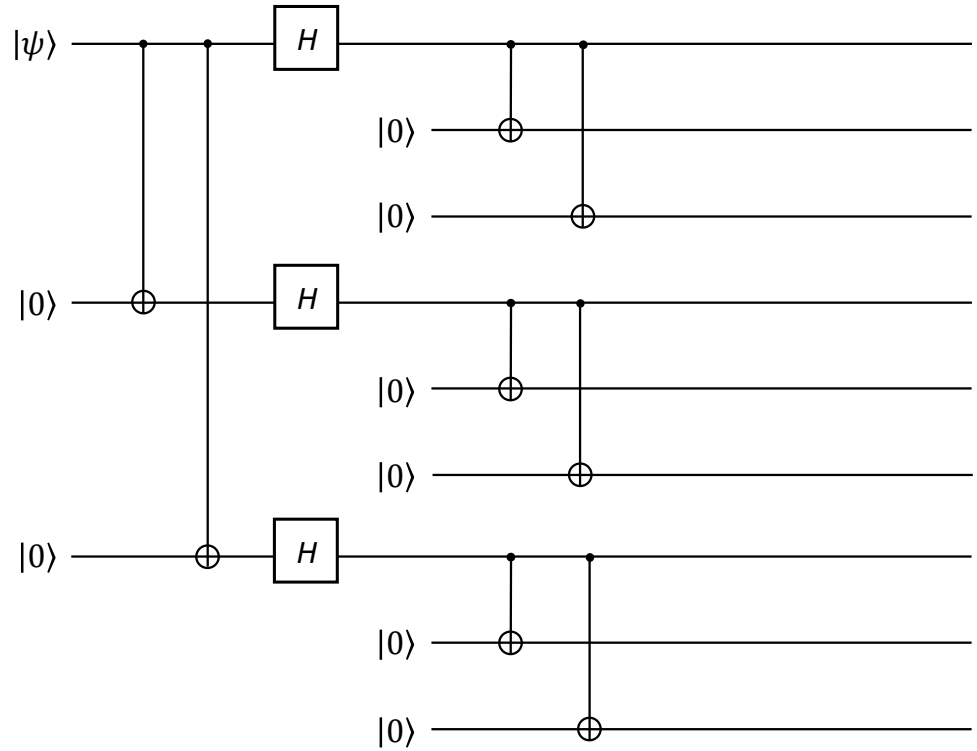
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

Thus we have:

Ph-flip	$\|\psi_1\rangle$	M_1	M_2	**Recovery**	$\|\psi_2\rangle$
-	$\alpha\|+++\rangle + \beta\|---\rangle$	0	0	$I \otimes I \otimes I$	$\alpha\|+++\rangle + \beta\|---\rangle$
1	$\alpha\|-++\rangle + \beta\|+--\rangle$	1	0	$Z \otimes I \otimes I$	$\alpha\|+++\rangle + \beta\|---\rangle$
2	$\alpha\|+-+\rangle + \beta\|-+-\rangle$	1	1	$I \otimes Z \otimes I$	$\alpha\|+++\rangle + \beta\|---\rangle$
3	$\alpha\|++-\rangle + \beta\|--+\rangle$	0	1	$I \otimes I \otimes Z$	$\alpha\|+++\rangle + \beta\|---\rangle$

The Shor code

The Shor code is a 9-qubit code which is constructed by *concatenating* the three-qubit bit-flip and three-qubit phase-flip codes:



This encodes the computational basis states as follows:

$$|0\rangle \to |0_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle \to |1_L\rangle = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Correcting bit-flips with the Shor code

The Shor code can detect and correct a bit-flip on any single qubit. For example, suppose we have an arbitrary quantum state $\alpha|0\rangle + \beta|1\rangle$ which we encode with the Shor code as:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

If a bit-flip occurs on the first qubit, the state becomes:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

Which can be detected (and thus recovered from) by parity-check measurements between the first three qubits as in the three-qubit bit-flip code. By symmetry we can see that the same principle applies to all of the nine qubits.

Correcting phase-flips with the Shor code

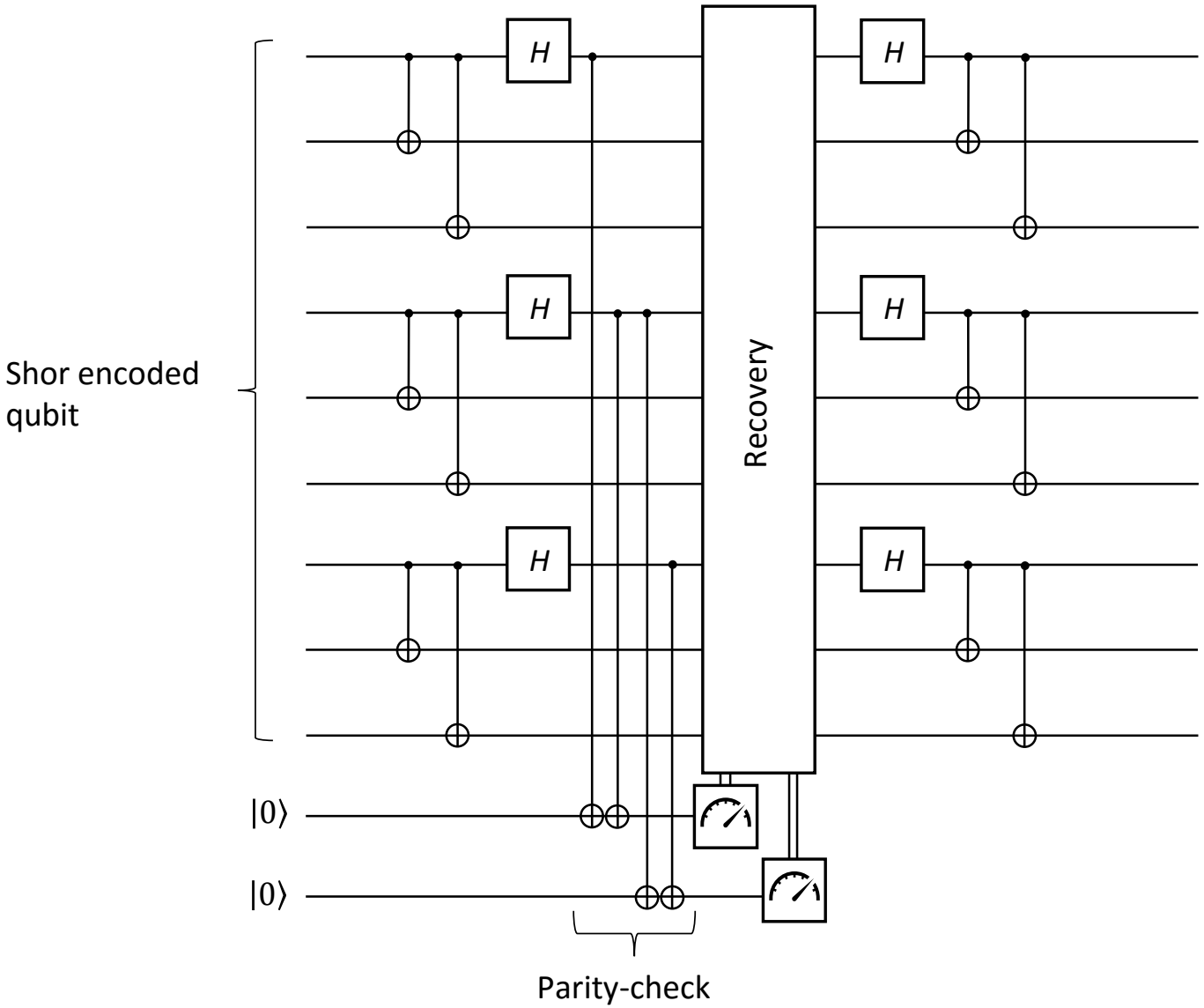
The Shor code can also detect and correct a phase-flip on any single qubit. If a phase-flip occurs on the first qubit, the state becomes:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

The key idea here is to detect which of the three blocks of three qubits has experienced a change of sign. This is achieved using the circuit shown on the following slide.

We can also correct combinations of bit- and phase-flips in this way.

Circuit for correcting phase-flips with the Shor code



The depolarising channel

When studying the (classical) three-bit repetition code, we saw that in practise it is more useful to think of it as a code that suppresses the error in the binary symmetric channel from p_e to $\mathcal{O}(p_e^2)$.

In the quantum case, we can see something similar: Consider the depolarising channel, in which a physical qubit is left unchanged with probability $1 - p_e$; experiences a bit-flip with probability $\frac{p_e}{3}$; experiences a phase-flip with probability $\frac{p_e}{3}$; or experiences both a bit- and phase-flip with probability $\frac{p_e}{3}$.

An analogous argument to that made for the binary symmetric channel can be made to show that the Shor code suppresses the error from p_e to $\mathcal{O}(p_e^2)$ in the depolarising channel.

Correcting any single qubit error with the Shor code (1)

Suppose the first qubit encounters an error which sends $|0\rangle \to a|0\rangle + b|1\rangle$ and $|1\rangle \to c|0\rangle + d|1\rangle$. We thus have the state:

$$\frac{1}{2\sqrt{2}}\Big(\alpha(a|000\rangle + b|100\rangle + c|011\rangle + d|111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(a|000\rangle + b|100\rangle - c|011\rangle - d|111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

Letting $k + m = a$, $k - m = d$, $l + n = b$ and $l - n = c$, we get

$$\begin{aligned}\frac{1}{2\sqrt{2}}\Bigg(&k\Big(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ &+ \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big) \\ &+l\Big(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ &+ \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big) \\ &+m\Big(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ &+ \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big) \\ &+n\Big(\alpha(|100\rangle - |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ &+ \beta(|100\rangle + |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\Bigg)\end{aligned}$$

Correcting any single qubit error with the Shor code (2)

As before, we first perform parity-check measurements to detect a bit-flip. The parity check for a bit-flip in the first block of three qubits requires two ancillas (the first comparing the first and second qubits, the second comparing the second and third qubits), whose state (after the parity-check CNOTs) we can append to the Shor code state:

$$\frac{1}{2\sqrt{2}}\Big(k\Big(\alpha(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle) + \beta(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\Big)|00\rangle + l\Big(\alpha(|100\rangle+|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle) + \beta(|100\rangle-|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\Big)|10\rangle + m\Big(\alpha(|000\rangle-|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle) + \beta(|000\rangle+|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\Big)|00\rangle + n\Big(\alpha(|100\rangle-|011\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle) + \beta(|100\rangle+|011\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)\Big)|10\rangle\Big)$$

Correcting any single qubit error with the Shor code (3)

If the parity-check measurement outcome is 00, the state collapses to (un-normalised):

$$k\Big(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$
$$+m\Big(\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$

In which case there is no bit-flip. Or if the measurement outcome is 10:

$$l\Big(\alpha(|100\rangle + |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|100\rangle - |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)$$
$$+n\Big(\alpha(|100\rangle - |011\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|100\rangle + |011\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)\Big)\Big)$$

i.e., a bit-flip has occurred which we can then correct.

Correcting any error by correcting only bit- and phase-flips

Following the bit-flip parity-check measurement (and correction if necessary) we perform a parity-check measurement to check for a phase flip. Using the same argument as for the bit-flip detection, if we measure 0 the state collapses to:

$$\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Or if we measure a 1 we get:

$$\alpha(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \beta(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

i.e., a phase-flip has occurred which we can then correct. Therefore we have recovered the original state.

Therefore performing bit- and phase-flip parity-check measurements collapses a general state into the case where either the bit / phase flip has occurred or not as per the measurement outcome. This remarkable property allows us to correct a continuum of errors by performing only bit- and phase-flip checks.

Digitisation of errors

Even though quantum errors are continuous, quantum error correction can be described as having the effect of *digitising* the errors, and thus we have digital (bit- and / or phase-flip) errors occurring with some probability. We note the following important further properties:

- When detecting a bit-flip, the probability of measuring $|00\rangle$ (i.e., no bit-flip) on the parity-check bits is $|k|^2 + |m|^2$.
- But recall that by definition the noise sends $|0\rangle \to a\,|0\rangle + b\,|1\rangle$ and $|1\rangle \to c\,|0\rangle + d\,|1\rangle$; and we have that $k = \frac{1}{2}(a + d)$.
- Thus we can see that for low noise levels ($a, d \approx 1$) we have a correspondingly high probability that no bit-flip occurs.
- The same principle applies to the detection of phase-flips.

The depolarising channel as a general noise model

In the example above we consider only the simple case in which a single qubit of the Shor code is subject to noise, and the remainder are left untouched. However, were we to perform a more detailed analysis in which each qubit is subject to noise, we would find that the same essential principle holds: if the noise on each qubit is small then it is likely that no bit- or phase-flip will have occurred.

Notably, the fact that error correction has this digitising effect on errors means that we can model general noisy channels as depolarising channels, given some mild assumptions (most importantly the independence of errors on the physical qubits that compose the code). That is, when the nine qubits that encode a single logical qubit using the Shor code are each corrupted by an independent general noisy channel, then we can analyse the errors as if each physical qubit were subject to depolarising noise.

Some noisy channels will be such that bit- or phase-flips are more likely, but it will usually be possible to make a conservative choice of depolarising channel model (i.e., not underestimating any source of error).

More sophisticated classical error correction

Repetition codes are useful for demonstrating the principle of error correction, but are rather too inefficient to use in practise. One particularly elegant code is the $(7, 4)$ Hamming code, a linear code that encodes a 4-bit *data-word*, d, as a 7-bit *code-word*, c, according to $c = Gd \mod 2$, where G is the generator matrix:

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Any errors are detected by applying the parity-check matrix, H, to a given code-word.

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Letting $p = Hc \mod 2$, the three bits of p are all zero if c is a valid code-word, and otherwise the seven possible three-bit strings with at least one 1 encode the position of a single error. Thus the $(7, 4)$ Hamming code can detect and correct any single bit error.

Quantum codes from classical codes

Classical *linear* codes are efficient, in the sense that code-words are generated by multiplying the data-word by a matrix, which can be compactly described. There is a technique for using classical linear codes to find quantum error correction codes. These codes are known as CSS (Calderbank-Shor-Steane) codes – although detailed analysis of these are beyond the scope of this course.

However, it is worth being aware of one particular CSS code, the *Steane code*, which is constructed from the $(7, 4)$ Hamming code and encodes the logical states 0 and 1 as follows:

$$|0_L\rangle = \frac{1}{\sqrt{8}}\big(\,|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$
$$|0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\,\big)$$
$$|1_L\rangle = \frac{1}{\sqrt{8}}\big(\,|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$
$$|1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle\,\big)$$

Like the Shor code, the Steane code guarantees to correct any bit- and / or phase-flip that occurs on a single qubit. Thus we can see that it also suppresses the error of the depolarising channel from p_e to $\mathcal{O}(p_e^2)$.

What to remember

We have seen that there are three obstacles to applying the techniques and principles of classical error correction directly to quantum error correction, each of which can be worked around:

- The no-cloning principle means that we cannot simply copy quantum states in repetition codes – instead we can use entangling to "copy" the information.
- Measurements destroy quantum information: so instead we design the error correcting codes so that the measurements only tell us whether an error has occurred, and nothing about the quantum state itself.
- Quantum errors are continuous: but we have seen that the process of error correction effectively digitises the errors.

Additionally, we have seen that, in practise, classical error correction codes are typically more sophisticated and efficient than simple repetition codes, and that these can be used to design quantum error correction codes, of which the Steane code is an important example.