Example:
Can we find $x, y, z$ such that

\[
\begin{align*}
x + y + z &\geq 4 \\
x - y &= 3 \\
z &\leq 2 \\
x &= 1
\end{align*}
\]
Constraint Satisfaction Problems

In general a constraint satisfaction problem (CSP) is specified by:

- A collection $V$ of variables.
- For each variable $x \in V$ a domain $D_v$ of possible values.
- A collection of constraints each of which consists of a tuple $(x_1, \ldots, x_r)$ of variables and a set
  $$S \subseteq D_{x_1} \times \cdots \times D_{x_r}$$
  of permitted combinations of values.

We consider finite-domain CSP, where the sets $D_x$ are finite.
We further make the simplifying assumption that there is a single domain $D$, with $D_x = D$ for all $x \in V$. 
Constraint Satisfaction Problems

In general a *constraint satisfaction problem (CSP)* is specified by:

- A collection $V$ of *variables*.
- A domain $D$ of *values*.
- A collection of *constraints* each of which consists of a tuple $(x_1, \ldots, x_r)$ of variables and a set $S \subseteq D^r$ of permitted combinations of values.

The problem is to *decide* if there is an assignment

$$\eta : V \to D$$

such that for each constraint $C = (x, S)$ we have

$$\eta(x) \in S.$$
Example - Boolean Satisfiability

Consider a Boolean formula $\phi$ in *conjunctive normal form* (CNF). This can be seen as *CSP* with

- $V$ the set of variables occurring in $\phi$
- $D = \{0, 1\}$
- a *constraint* for each *clause* of $\phi$.

The clause $x \lor y \lor \overline{z}$ gives the constraint $(x, y, z), S$ where

$$S = \{(0, 0, 0), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$$
Fix a relational signature $\sigma$ (no function or constant symbols). Let $A$ and $B$ be two $\sigma$-structures. A *homomorphism* from $A$ to $B$ is a function $h : A \rightarrow B$ such that for each relation $R \in \sigma$ and each tuple $a$

$$a \in R^A \Rightarrow h(a) \in R^B$$

The problem of deciding, given $A$ and $B$ whether there is a homomorphism from $A$ to $B$ is NP-complete. Why?
Homomorphism and CSP

Given a CSP with variables $V$, domain $D$ and constraints $C$, let $\sigma$ be a signature with a relation symbol $R_S$ of arity $r$ for each distinct relation $S \subseteq D^r$ occurring in $C$.

Let $\mathcal{B}$ be the $\sigma$-structure with universe $D$ where each $R_S$ is interpreted by the relation $S$.

Let $\mathcal{A}$ be the structure with universe $V$ where $R_S$ is interpreted as the set of all tuples $x$ for which $(x, S) \in C$.

Then, the CSP is solvable if, and only if, there is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.
Write $A \rightarrow B$ to denote that *there is* a homomorphism from $A$ to $B$.

The problem of determining, given $A$ and $B$, whether $A \rightarrow B$ is *NP-complete*, and can be decided in time $O(|B|^{|A|})$.

So, for a fixed structure $A$, the problem of deciding membership in the set

$$\{B \mid A \rightarrow B\}$$

is in $P$. 
Non-uniform CSP

On the other hand, for a fixed structure $B$, we define the non-uniform CSP with template $B$, written $CSP(B)$ as the class of structures

$$\{A \mid A \rightarrow B\}$$

The complexity of $CSP(B)$ depends on the particular structure $B$. The problem is always in NP. For some $B$, it is in P and for others it is NP-complete.
Example - 3-SAT

Let $\mathbb{B}$ be a structure with universe $\{0, 1\}$ and *eight* relations

$$R_{000}, R_{001}, R_{010}, R_{011}, R_{100}, R_{101}, R_{110}, R_{111}$$

where $R_{ijk}$ is defined to be the relation

$$\{0, 1\}^3 \setminus \{(i, j, k)\}.$$

Then, $\text{CSP}(\mathbb{B})$ is *essentially* the problem of determining satisfiability of Boolean formulas in *3-CNF*.
Example - 3-Colourability

Let $K_n$ be the *complete* simple undirected graph on $n$ vertices.

Then, an undirected simple graph is in CSP($K_3$) *if, and only if*, it is 3-colourable.

CSP($K_3$) is NP-complete.

On the other hand, CSP($K_2$) is in P.
Example - 3XOR-SAT

Let $\mathcal{B}$ be a structure with universe $\{0, 1\}$ and two ternary relations $R_0$ and $R_1$. 

where $R_i$ is the collection of triples $(x, y, z) \in \{0, 1\}^3$ such that 

$$x + y + z \equiv i \pmod{2}$$

Then, $\text{CSP}(\mathcal{B})$ is essentially the problem of determining satisfiability of Boolean formulas in 3-XOR-CNF. This problem is in $\mathsf{P}$. 
Schaefer’s theorem

Schaefer (1978) proved that if \( B \) is a structure on domain \( \{0, 1\} \), then \( \text{CSP}(B) \) is in \( P \) if one of the following cases holds:

1. Each relation of \( B \) is \textit{0-valid}.
2. Each relation of \( B \) is \textit{1-valid}.
3. Each relation of \( B \) is \textit{bijunctive}.
4. Each relation of \( B \) is \textit{Horn}.
5. Each relation of \( B \) is \textit{dual Horn}.
6. Each relation of \( B \) is \textit{affine}.

In all other cases, \( \text{CSP}(B) \) is \textit{NP-complete}.
Let $H$ be a \textit{simple, undirected graph}.

\textbf{Hell and Nešetřil (1990)} proved that $\text{CSP}(H)$ is in P if one of the following holds

1. $H$ is \textit{edgeless}
2. $H$ is \textit{bipartite}

In all other cases, $\text{CSP}(H)$ is \textit{NP-complete}.
Feder and Vardi (1993) conjectured that for every finite relational structure $\mathcal{B}$:

either $\text{CSP}(\mathcal{B})$ is in $\text{P}$ or it is $\text{NP}$-complete.

Ladner (1975) showed that for any languages $L$ and $K$, if $L \leq_P K$ and $K \not\leq_P L$, then there is a language $M$ with

$$L \leq_P M \leq_P K \text{ and } K \not\leq_P M \text{ and } M \not\leq_P L$$

Corollary: if $P \neq \text{NP}$ then there are problems in $\text{NP}$ that are neither in $P$ nor $\text{NP}$-complete.
Bulatov and Zhuk (2017) independently proved the Feder-Vardi dichotomy conjecture.

The result came after a twenty-year development of the theory of CSP based on universal algebra.

The complexity of \( \text{CSP}(\mathcal{B}) \) can be completely classified based on the identities satisfied by the algebra of polymorphisms of the structure \( \mathcal{B} \).