Foundations of Computer Science
Lecture 7:
Dictionaries and Functional Arrays

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22nd October 2021

Warmup

```
# type 'a tree =
  | Br of 'a * 'a tree * 'a tree
  | ??
```

What's the missing definition here to make a binary tree?

Warmup

```
# type 'a tree =
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  | Lf
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What is the term when **'a** is present in the type definition?

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polymorphic

What is the term when the type definition refers to itself?

recursive

Dictionaries

- A dictionary attaches **values** to identifiers (known as **keys**).
- Define the **operations** we want over the dictionary:
  - **`lookup`** : find an item in the dictionary
  - **`update`** / `insert` : replace / store an item in the dictionary
  - `delete` : remove an item from the dictionary
  - `empty` : the null dictionary with no keys
  - **`Missing`** : exception for errors in lookup and delete

Implementing a dictionary

- Simplest representation for a dictionary is an **association list** (a list of key/value tuples).

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  | (x, y) :: pairs, a ->
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      else
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val lookup : ('a * 'b) list * 'a -> 'b = <fun>
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Lookup is O(n)

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Lookup is O(n)

Update is O(1)

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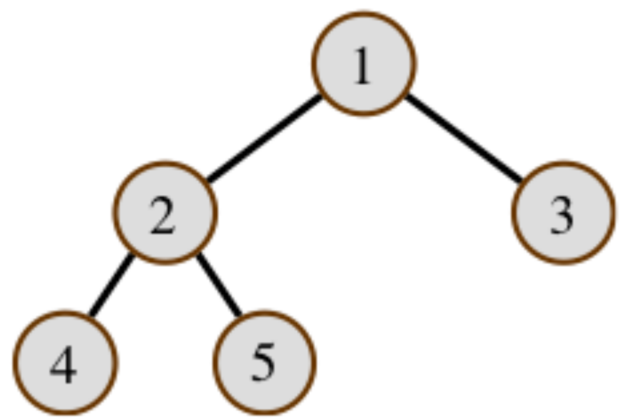
Lookup is O(n)

Update is O(1)

But what is the space usage?

Binary Search Trees

- Use **binary trees** as a more efficient representation than a list to get a better lookup complexity.

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Binary Search Trees

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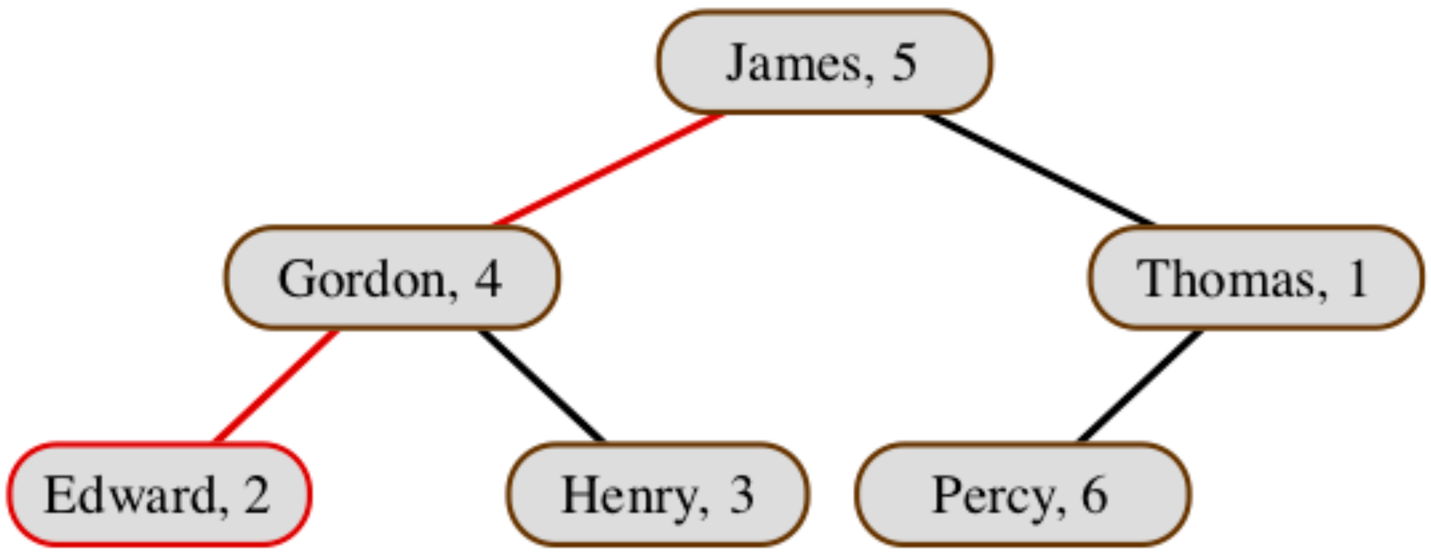

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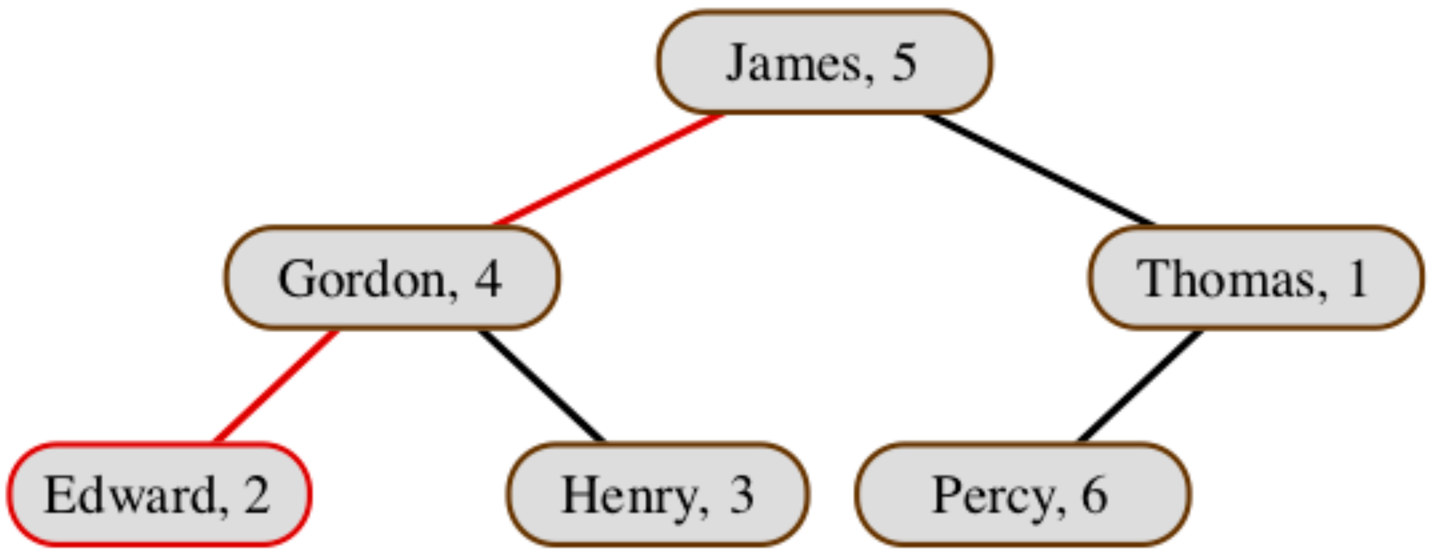

- Each node holds a `(key, value)` with a total ordering for the keys
- The *left* subtree holds smaller keys and the *right* subtree holds larger keys

Binary Search Trees

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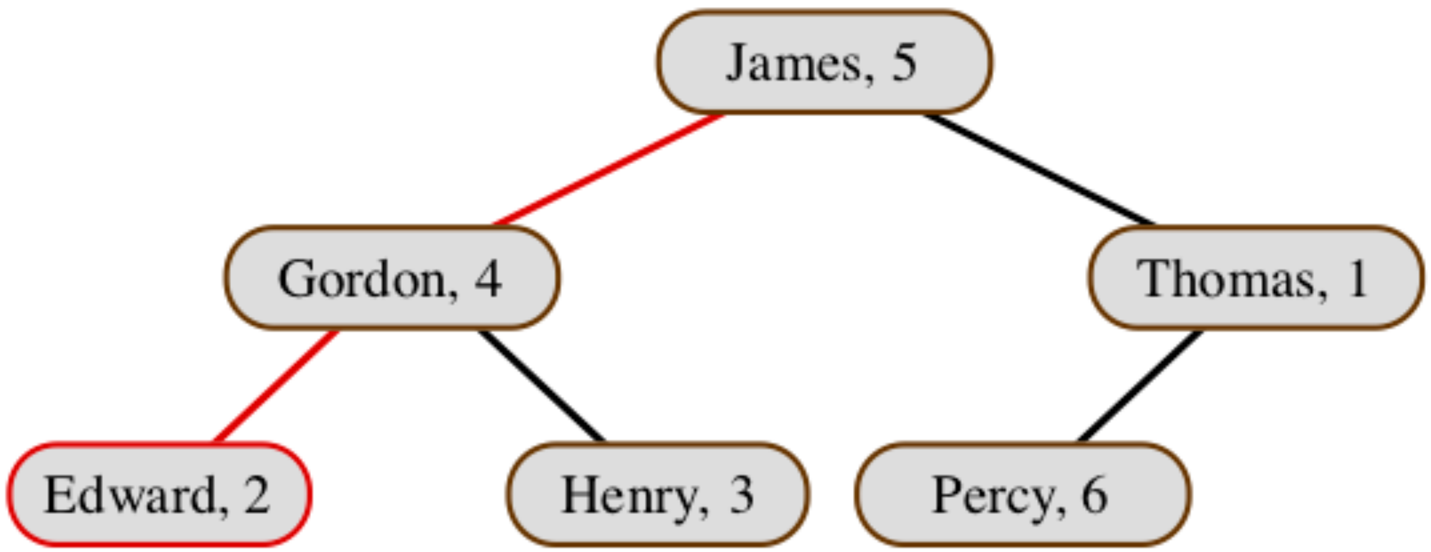

- If *balanced* then lookup is $O(\log n)$

- If *unbalanced* then lookup can be $O(n)$

Binary Search Trees

```
# exception Missing of string
exception Missing of string

# let rec lookup = function
  | Br ((a, x), t1, t2), b ->
      if b < a then
        lookup (t1, b)
      else if a < b then
        lookup (t2, b)
      else
        x
  | Lf, b -> raise (Missing b)
val lookup : (string * 'a) tree * string -> 'a = <fun>
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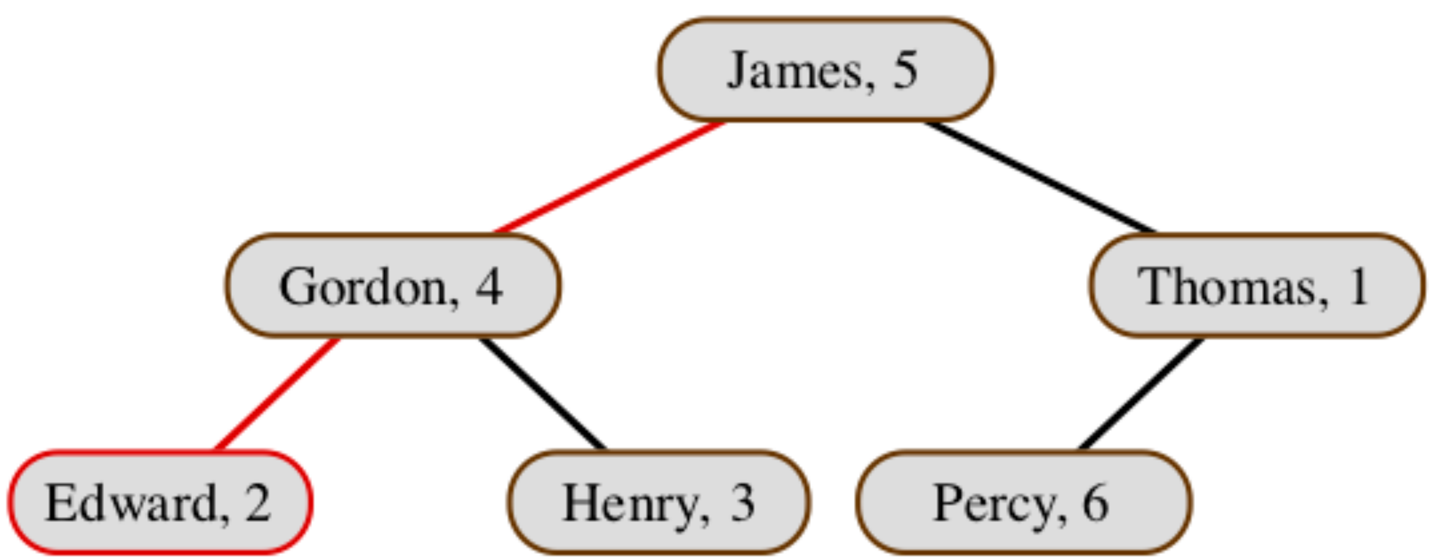

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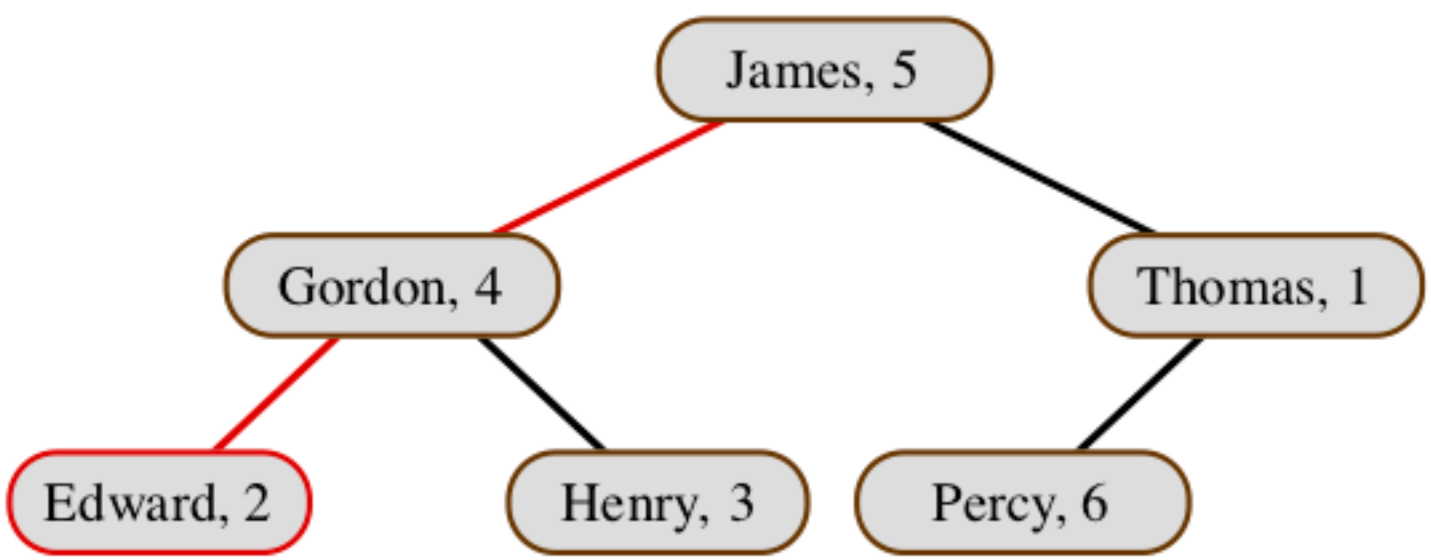

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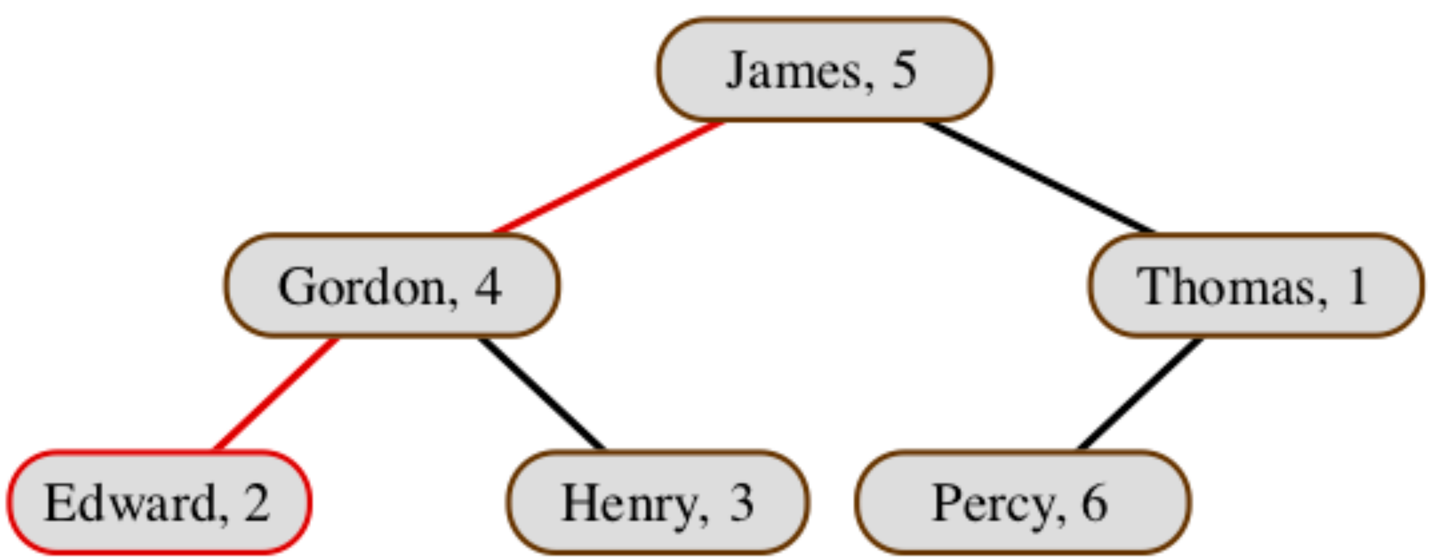
O(log n) if the tree is balanced



Binary Search Trees

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# let rec update k v = function
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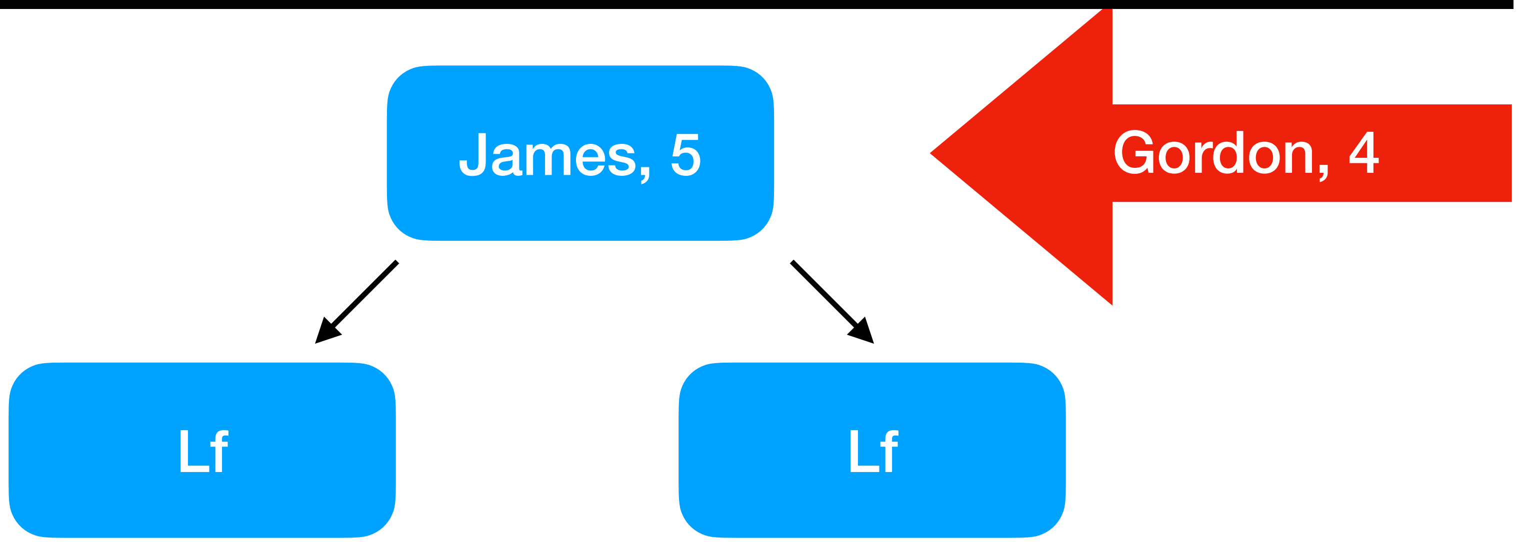

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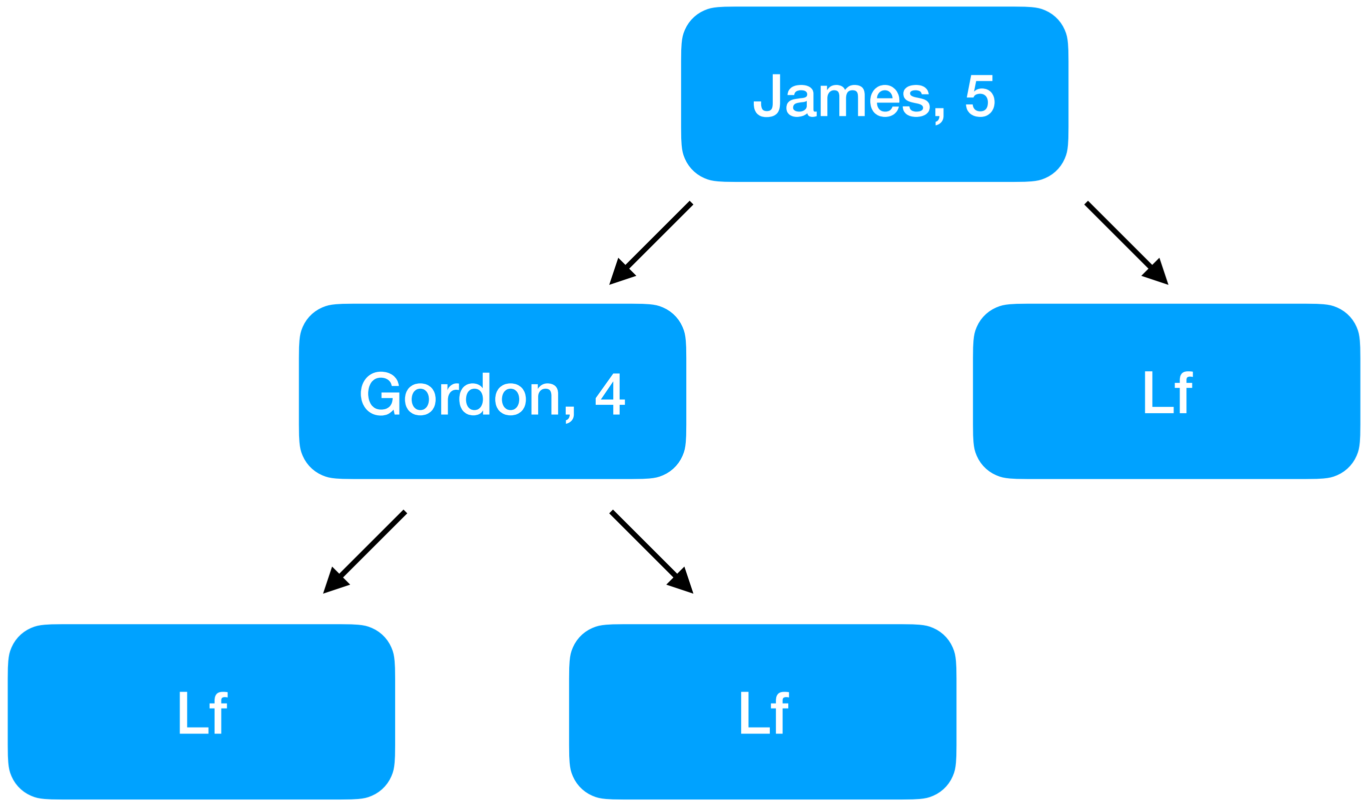

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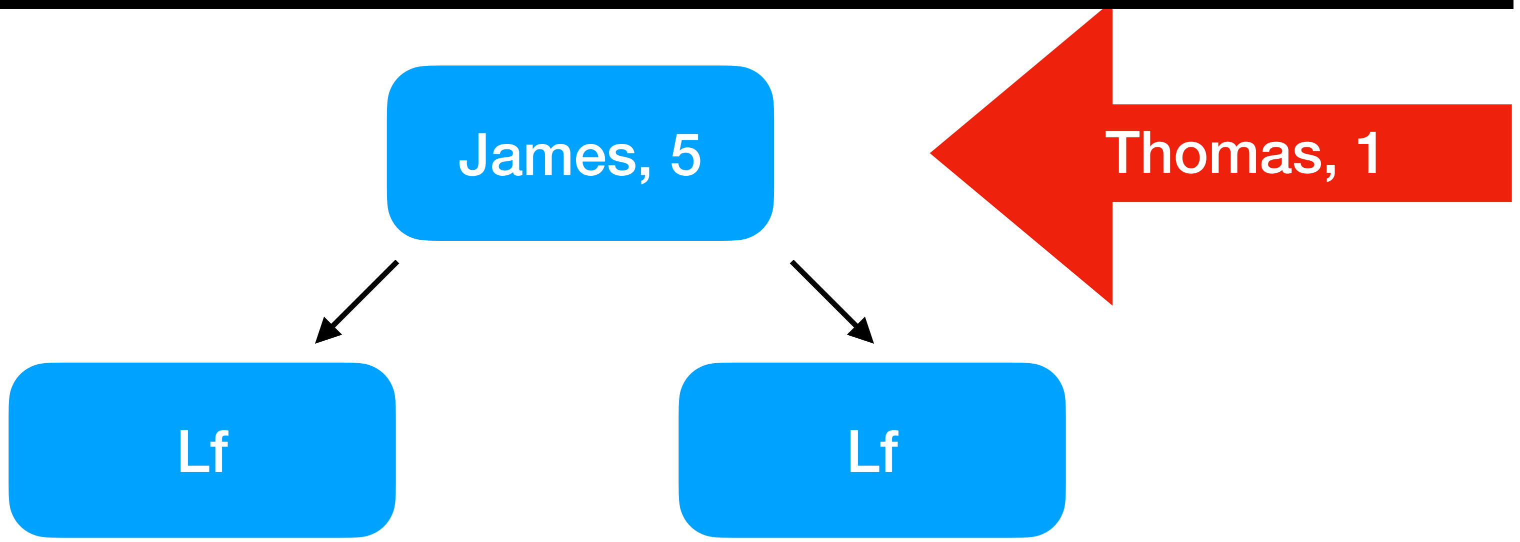

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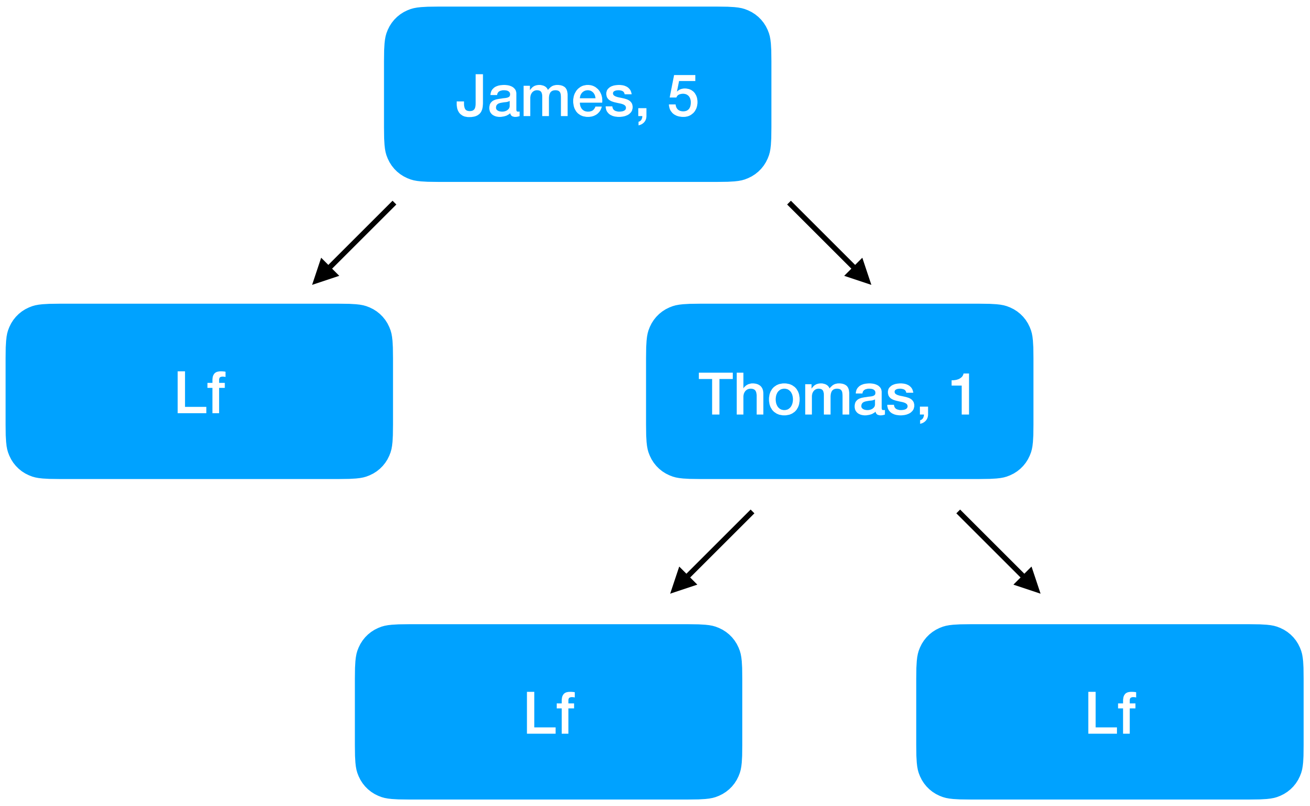

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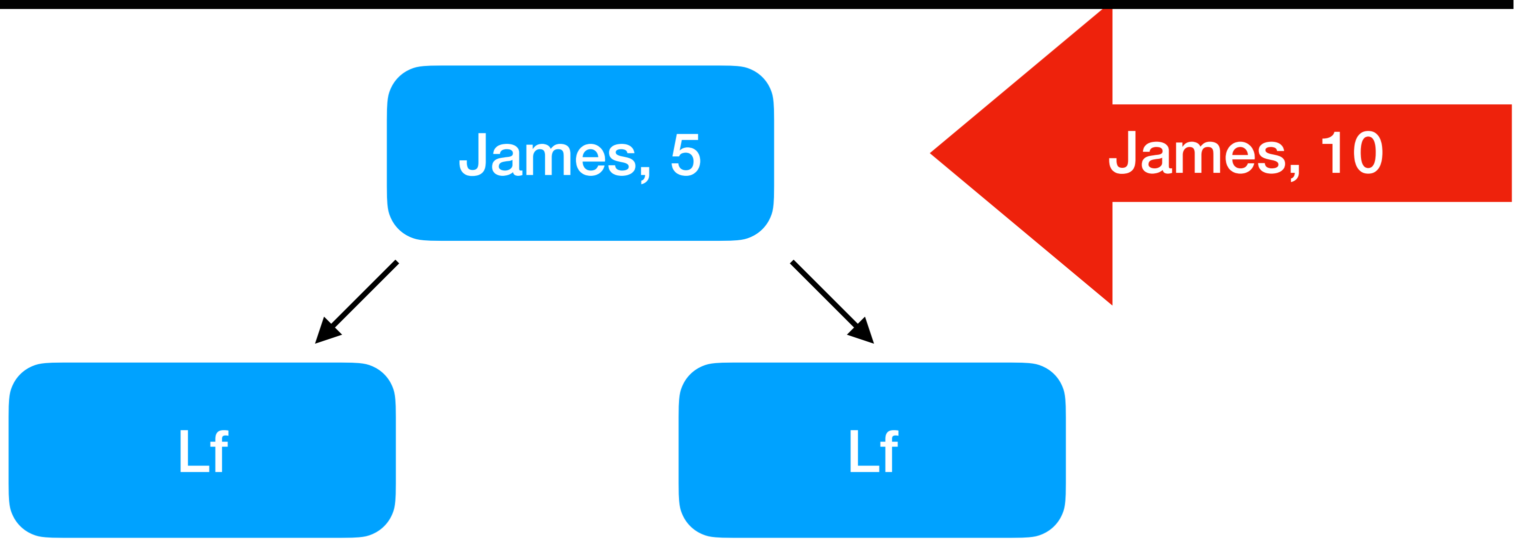

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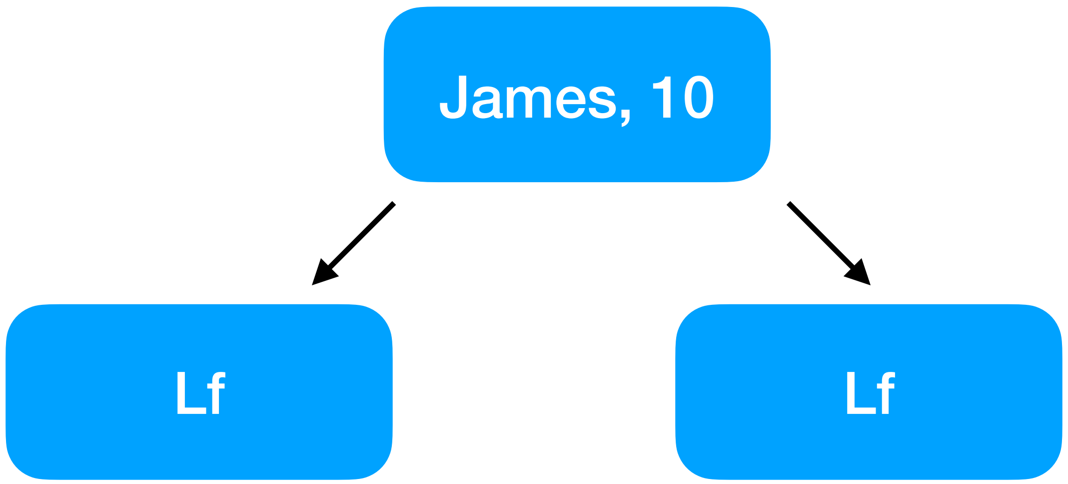

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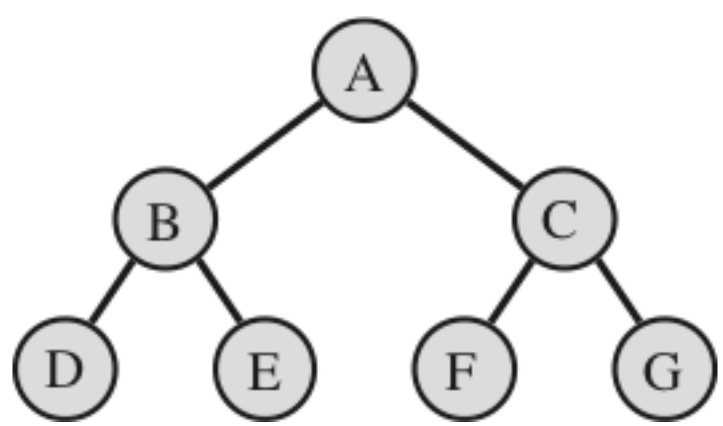
- We *reconstruct* the part of the structure that has changed and return the *updated* version.
- OCaml *shares* the original structure, and values pointing to the original remain unchanged.
- This is also known as a *persistent data structure.*

Traversing Trees

Tree traversal refers to visiting the nodes of each tree in a well-defined order.

- **preorder** visits the label first (`ABDECFG`)
- **inorder** visits the label midway (`DBEAFCG`)
- **postorder** visits the label last (`DEBFGCA`)



Traversing Trees: preorder

- **preorder** visits the label first (`ABDECFG`)

```
# let rec preorder = function
  | Lf -> []
  | Br (v, t1, t2) ->
      [v] @ preorder t1 @ preorder t2

val preorder : 'a tree -> 'a list = <fun>
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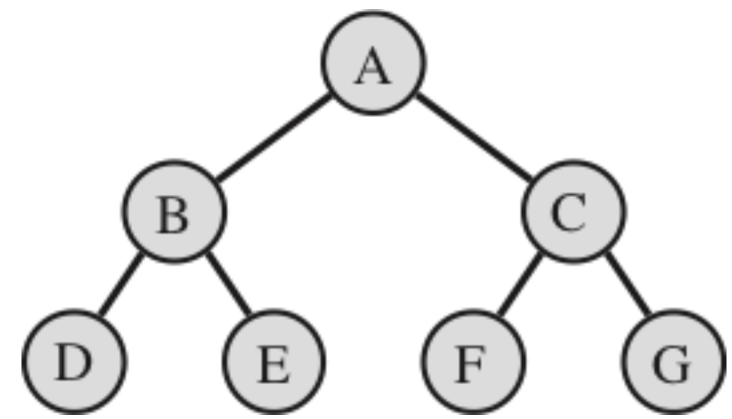

Traversing Trees: inorder

- **inorder** visits the label midway (DBEAFCG)

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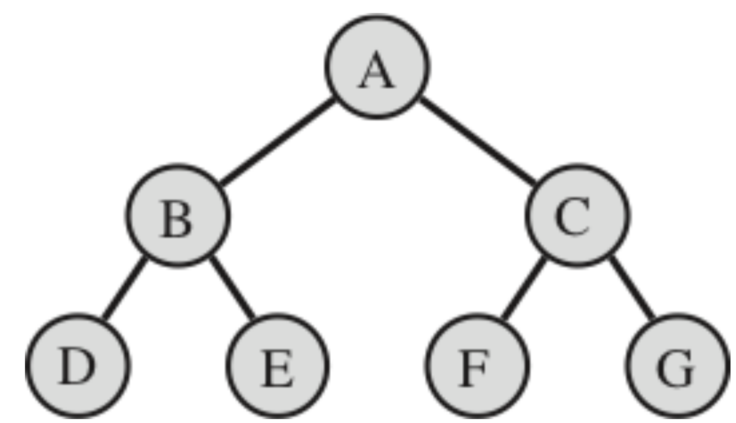

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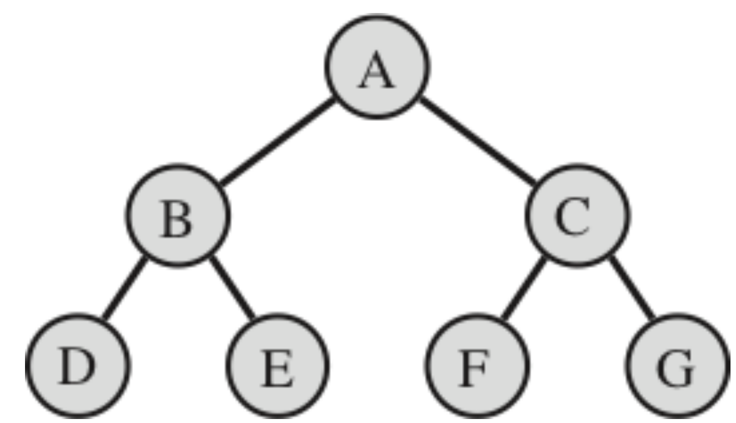

For binary search trees, this order respects the sorting constraint (left key < right key)

Also imaginatively known as a **treesort**.

Traversing Trees: postorder

- **postorder** visits the label last (`DEBFGCA`)

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# let rec postorder = function
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  | Br (v, t1, t2) ->
      postorder t1 @ postorder t2 @ [v]

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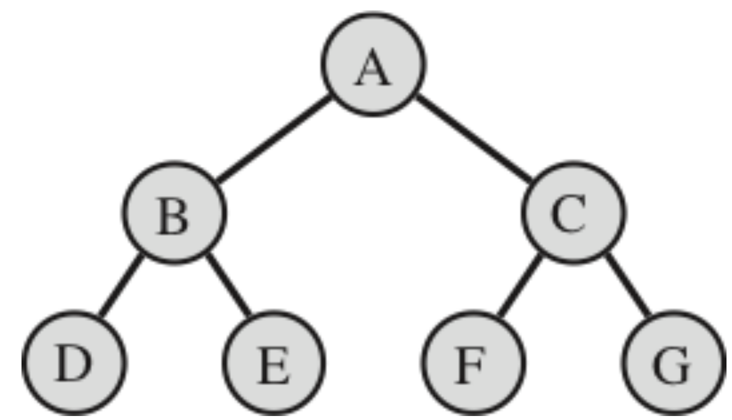

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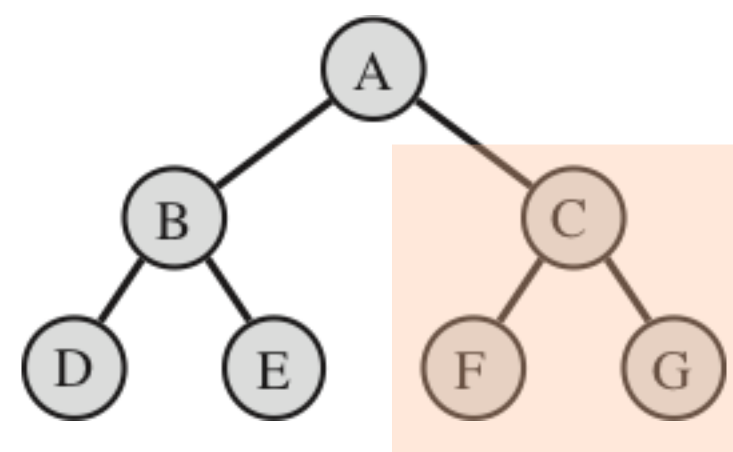

FGC

Traversing Trees: postorder

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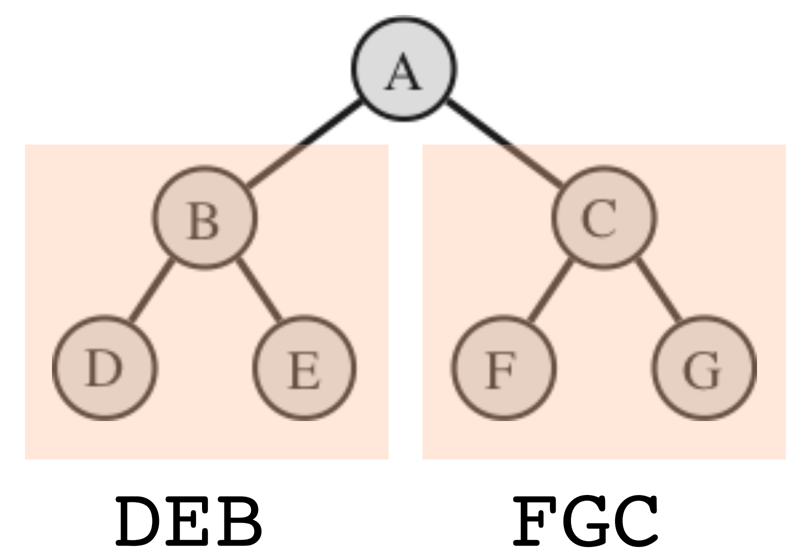

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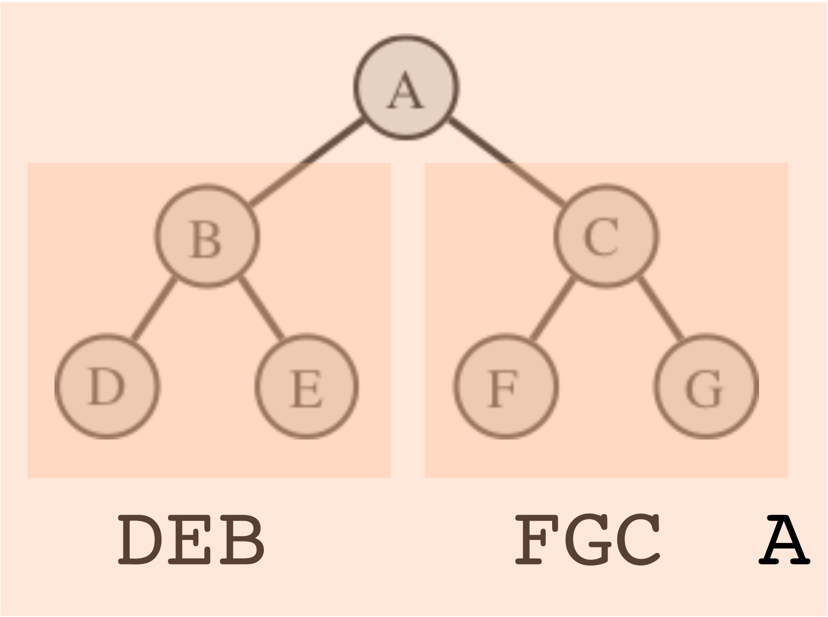
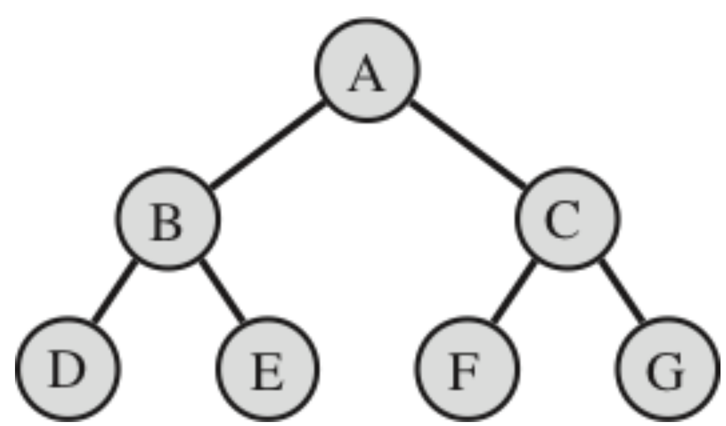


Traversing Trees

Tree traversal refers to visiting the nodes of each tree in a well-defined order.

- preorder, inorder and postorder are **depth-first** traversal algorithms.
- The other possibility is **breadth-first** by going across the levels of the tree.



Arrays

Arrays are an indexed storage area for values

- Very common data structure alongside lists and trees in most languages.
- Arrays are usually updated *in-place* and are *imperative* or *mutable* data structures.
- Are used in many classic algorithms such as the original Hoare in-place partition-sort.

Arrays

Arrays are an indexed storage area for values

- Elements of a list can only be reached by counting from the head of the list.
- Elements of a tree can be reached by following a path from the root.
- Elements of an array are uniformly designated by number (the "subscript").

Functional Arrays

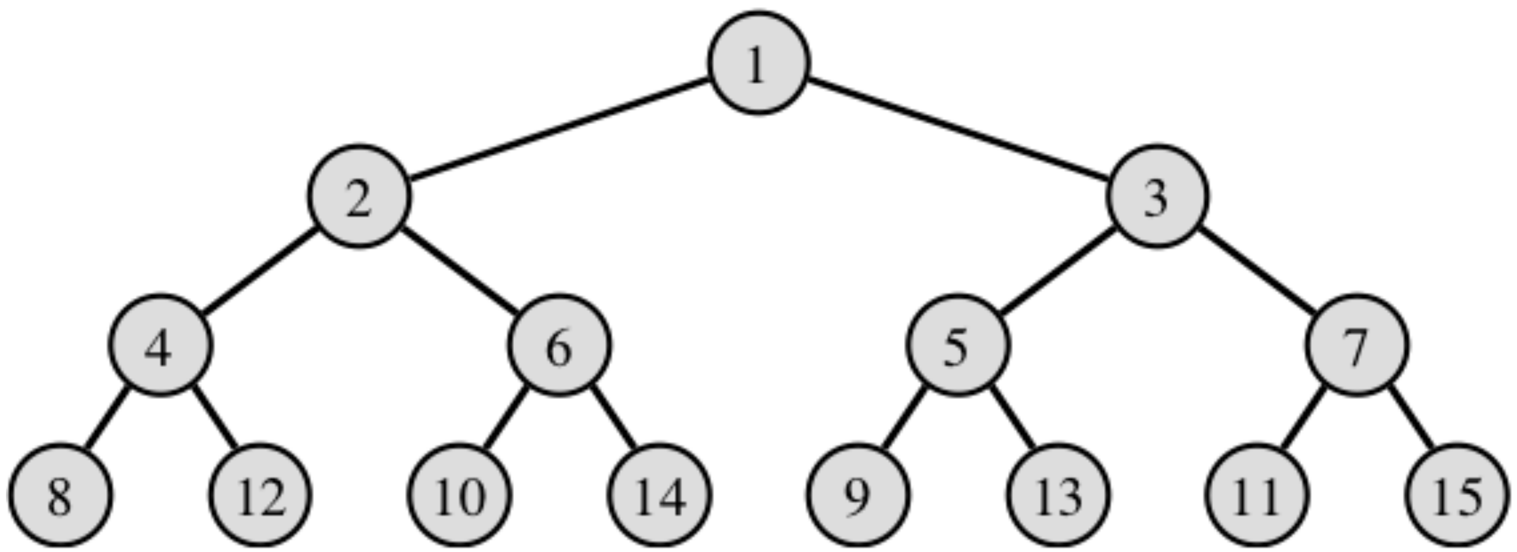
Arrays are an indexed storage area for values

Let's first consider an immutable array

- This is known as a *functional array* that is a finite map from integers to data.
- Updating implies copying the array to return a new version, but pointers to old copies remain.
- Can updates be efficient?

Functional Trees

The path to element *i* follows the **binary code** for *i* (the "subscript")



- The numbers above are not the values, but the positions of array elements.
- Complexity of access to this is always $O(\log n)$ as the tree is always balanced.

Functional Trees

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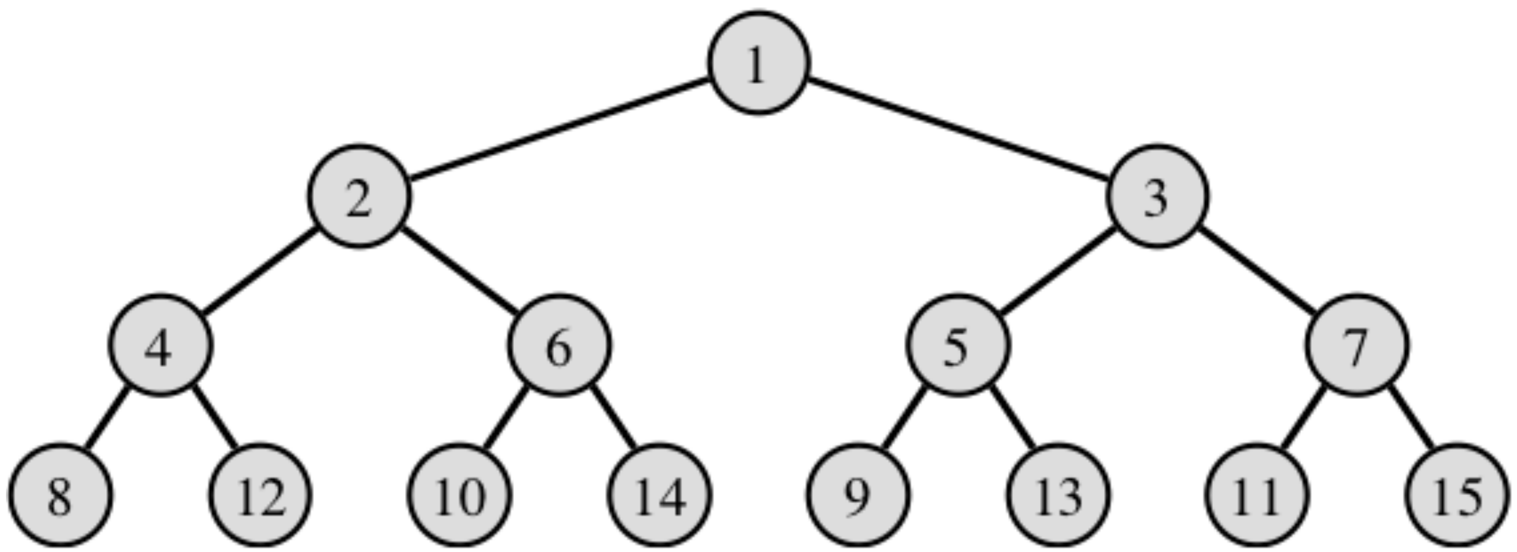
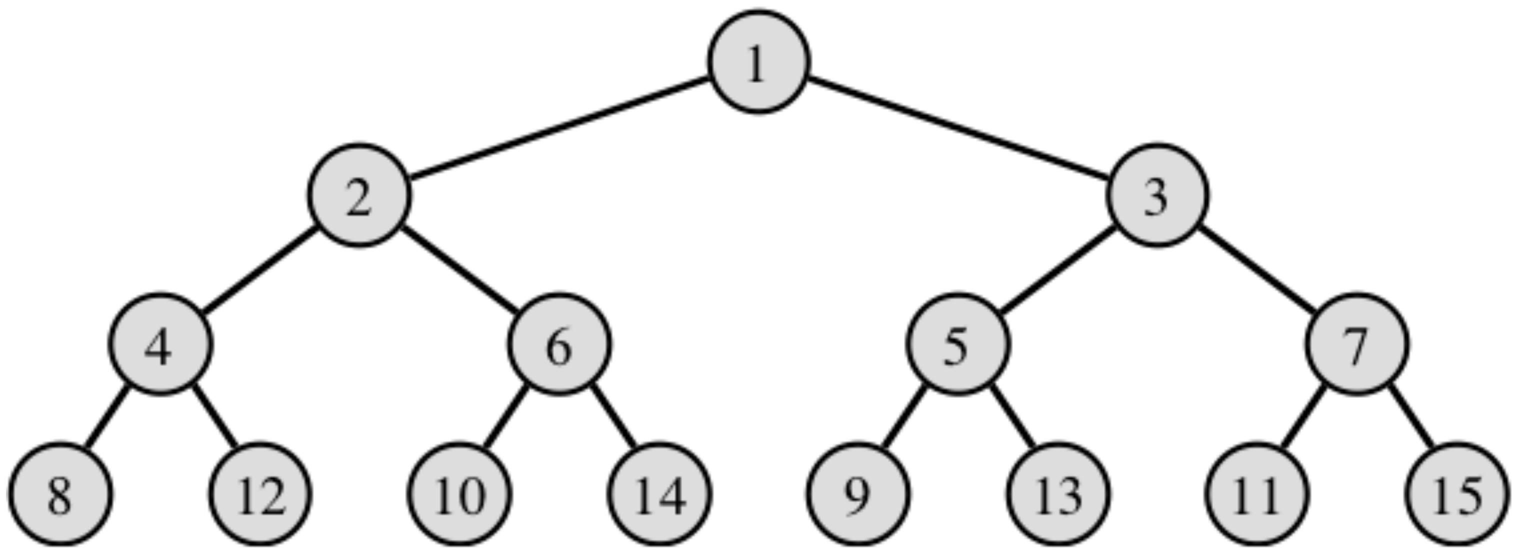



```
# exception Subscript

# let rec sub = function
  | Lf, _ -> raise Subscript
  | Br (v, t1, t2), k ->
      if k = 1 then v
      else if k mod 2 = 0 then
        sub (t1, k / 2)
      else
        sub (t2, k / 2)
```

Functional Trees

The path to element *i* follows the **binary code** for *i* (the "subscript")

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# let rec sub = function
  | Lf, _                                  -> raise Subscript
  | Br (v, t1, t2), 1                      -> v
  | Br (v, t1, t2), k when k mod 2 = 0    -> sub (t1, k / 2)
  | Br (v, t1, t2), k                      -> sub (t2, k / 2)
```

Functional Trees

The path to element *i* follows the **binary code** for *i* (the "subscript")

```
# let rec update = function
  | Lf, k, w ->
      if k = 1 then
        Br (w, Lf, Lf)
      else
        raise Subscript  (* Gap in tree *)
  | Br (v, t1, t2), k, w ->
      if k = 1 then
        Br (w, t1, t2)
      else if k mod 2 = 0 then
        Br (v, update (t1, k / 2, w), t2)
      else
        Br (v, t1, update (t2, k / 2, w))
```

Functional Trees

The path to element *i* follows the **binary cod** script")

O(log n) if the tree is balanced

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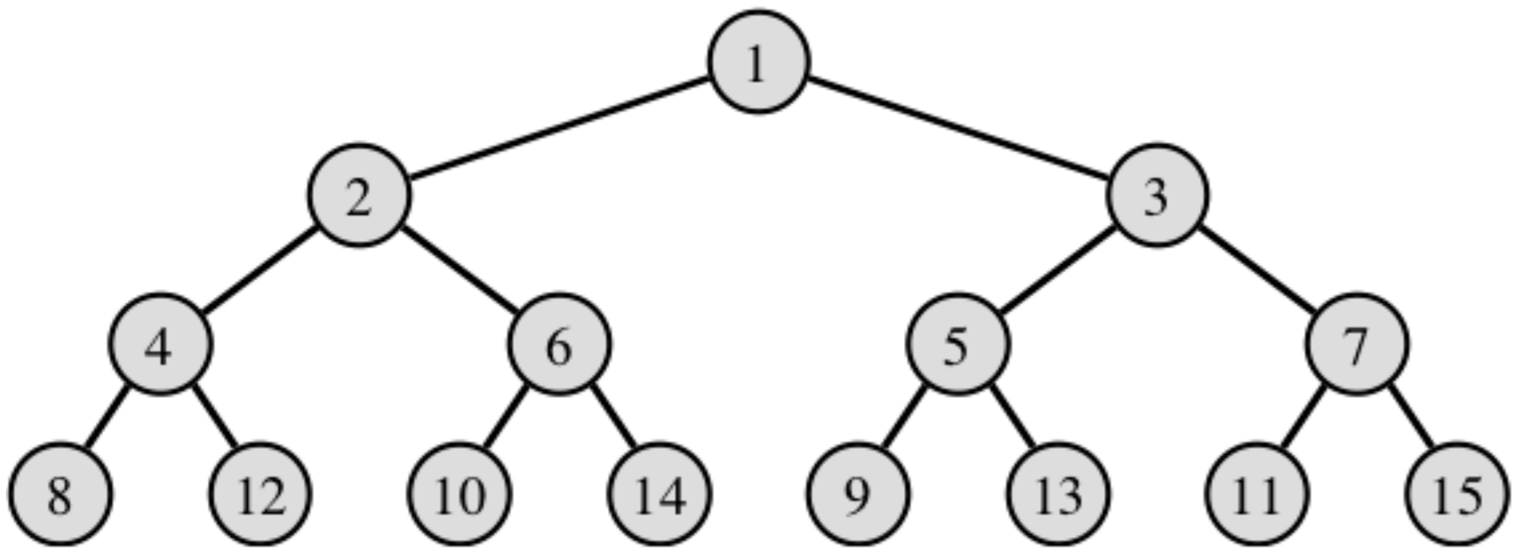

15 = 0b1111

12 = 0b1100

11 = 0b1011

Complexity of Dictionary Data Structures

- **Linear search:** Most general, needing only equality on keys, but inefficient (linear time).

- **Binary search:** Needs an ordering on keys.
  $O(\log n)$in the average case,
  binary search trees are $O(n)$ in the worst case.

- **Array subscripting:** Least general, requiring keys to be integers, but even worst-case time is $O(\log n)$.