Equivalence relations and set partitions

- Equivalence relations.

$R \subseteq A \times A$ is an equivalence relation whenever

(1) Reflexive: $\forall x \in A.\ x R x$

(2) Symmetry: $\forall x, y \in A.\ x R y \Rightarrow y R x$

(3) Transitive: $\forall x, y, z \in A.$

$$x R y \wedge y R z \Rightarrow x R z.$$

Examples:

- For m a positive integer, let $R_m \subseteq \mathbb{Z} \times \mathbb{Z}$

  $x \, R_m \, y \overset{\text{def}}{\iff} x \equiv y \pmod{m}$

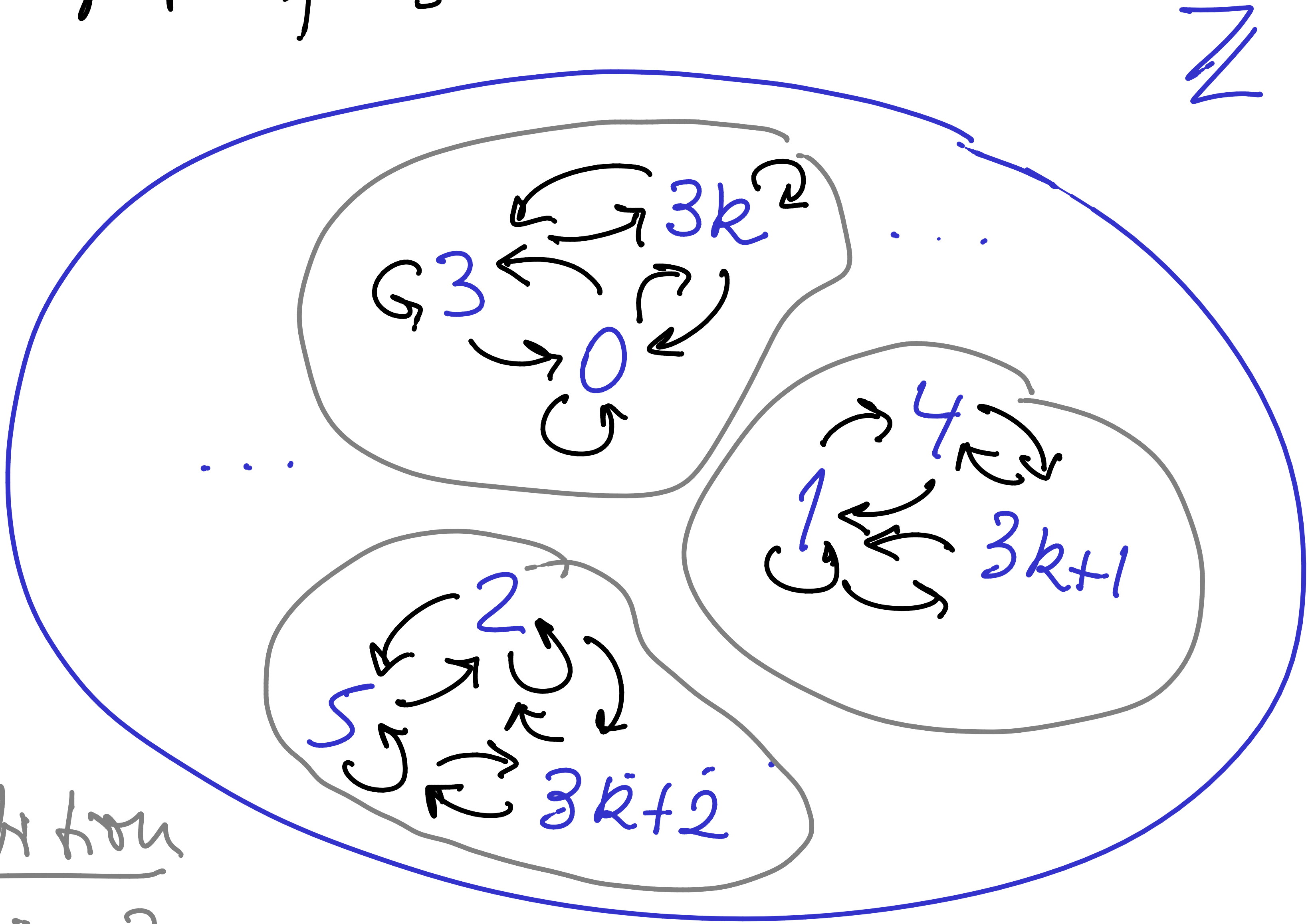
- Let A be a set.

  $I \subseteq \mathcal{P}(A) \times \mathcal{P}(A)$

  $\overset{\text{def}}{=}$

  $\{ (U, V) \in \mathcal{P}(A) \times \mathcal{P}(A) \mid U \cong V \}$.

Internal graph of R_3



A partition of $\mathbb{Z}$ in 3 equivalence classes.

- Set partitions.

A partition P of a set A is a set of subsets of A $\quad P \subseteq \mathcal{P}(A)$

such that

(1) $\emptyset \notin P$

(2) $\bigcup P = A$

(3) $\forall u, v \in P.\ u \neq v \Rightarrow u \cap v = \emptyset$

Examples: Partitions of $\mathbb{Z}$.

$P_1 = \{ \mathbb{Z} \}$

$P_2 = \{ \text{Odd}, \text{Even} \}$

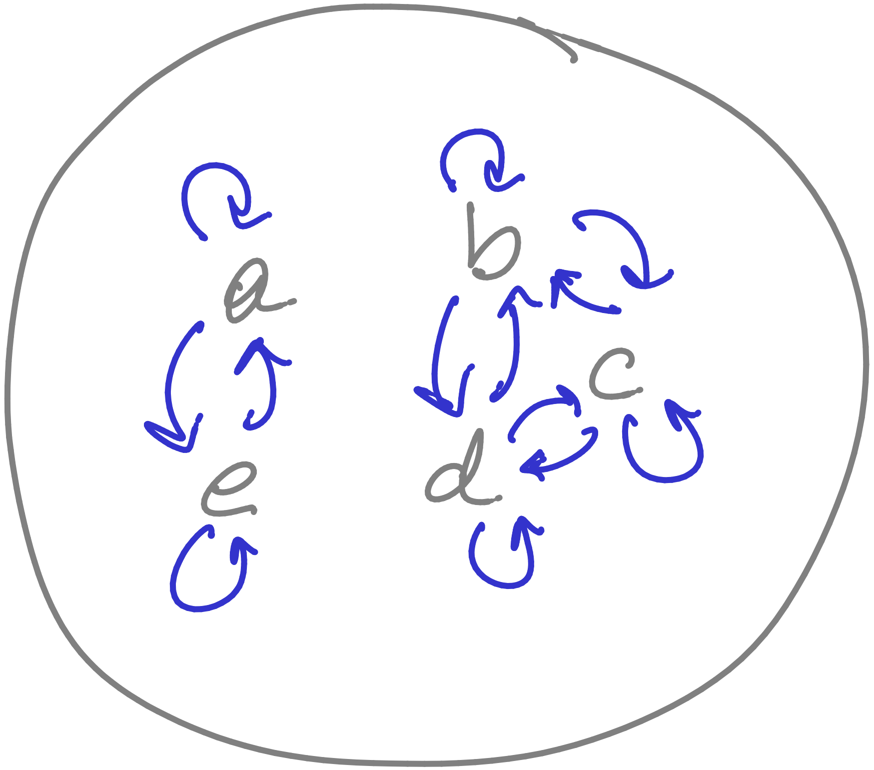
$P_3 = \{ \{3k \mid k \in \mathbb{Z}\}, \{3k+1 \mid k \in \mathbb{Z}\}, \{3k+2 \mid k \in \mathbb{Z}\} \}$

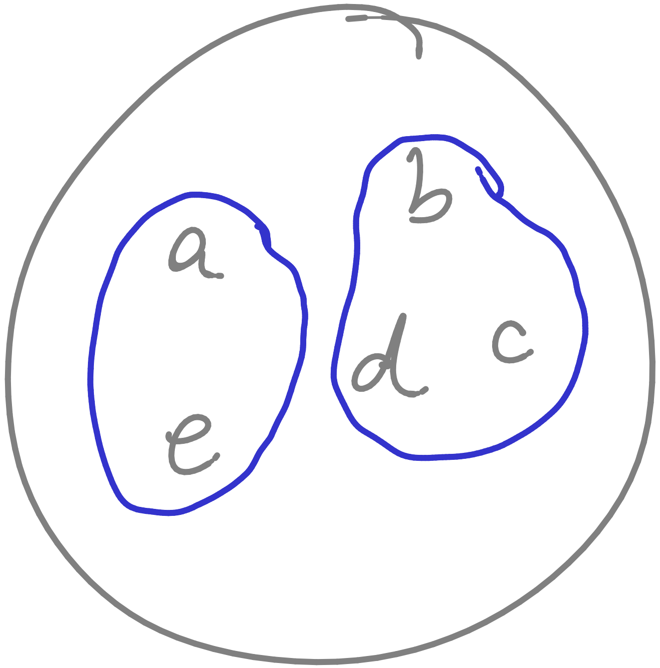
Theorem 134 *For every set A,*

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A) .$$

PROOF: all equivalence relations on A

all partitions of A






a
b
c
d
e
a
b
c
d
e

$$\underline{\mathrm{part}} : \underline{\mathrm{EqRel}}(A) \longrightarrow \underline{\mathrm{Part}}(A)$$

$$E \longmapsto \underline{\mathrm{part}}(E)$$

Def

$$\underline{\mathrm{part}}(E) \subseteq \mathcal{P}(A)$$

$$\| $$

$$\{ \beta \subseteq A \mid \exists a \in A.\ \beta = [a]_E \}$$

where

$$[a]_E = \{ x \in A \mid x \mathrel{E} a \}$$ Equivalence class of a

RTP: $\underline{\mathrm{part}}(E)$ is a partition.

(1) Every block in $\underline{part}(E)$ is non-empty.

Because $a \in [a]_E$

(2) $\bigcup \underline{part}(E) = A$

($\subseteq$) Clear.

($\supseteq$) Every element of A appears in a block.

$\forall a \in A. \exists \beta \in \underline{part}(E)$ namely $\beta = [a]_E$

such that $a \in \beta$.

(3) $\underline{RTP}$ If $\beta_1, \beta_2 \in \underline{part}(E)$ such that $\beta_1 \cap \beta_2 \neq \emptyset$

then $\beta_1 = \beta_2$

Let $\beta_1, \beta_2 \in \underline{\text{part}}(E)$ such that $\beta_1 \cap \beta_2 \neq \emptyset$

Then $\beta_1 = [a_1]_E$ for some $a_1 \in A$.

$\beta_2 = [a_2]_E$ for some $a_2 \in A$.

Also $b \in [a_1]_E$ and $b \in [a_2]_E$ for some b.

$$b \in [a_1]_E \iff b \, E \, a_1 \iff a_1 \, E \, b \quad (1)$$

$$(2) \quad b \, E \, a_2 \iff b \in [a_2]_E$$

$$(1) \wedge (2) \Rightarrow \boxed{a_1 \, E \, a_2 \iff [a_1]_E = [a_2]_E}$$

Lemma (exercise)

$\boxtimes$

$$\underline{eq} : \underline{Part}(A) \longrightarrow \underline{EqRel}(A)$$

$$P \longmapsto \underline{eq}(P)$$

Def: $\underline{eq}(P) \subseteq A \times A$

$$\underline{eq}(P) = \{ (x,y) \in A \times A \mid \exists \beta \in P.\ x \in \beta \wedge y \in \beta \}$$

RTP: $\underline{eq}(P)$ is an equivalence relation.

(1) $\forall a \in A.\ (a,a) \in \underline{eq}(P)$

$\iff \forall a \in A.\ \exists \beta \in P.\ a \in \beta.$ $\Longleftrightarrow$ $\bigcup P \supseteq A$

(2) $\forall x,y \in A.\ x\ \underline{eq}(P)\ y \Rightarrow y\ \underline{eq}(P)\ x$

Trivial.

(3) $\forall x, y, z \in A.$

$$x \; eq(P) \; y \wedge y \; eq(P) \; z \overset{?}{\Longrightarrow} x \; eq(P) \; z$$

$\exists \beta \in P.\ x \in \beta \wedge y \in \beta$ ① $\Longrightarrow$ $y \in \beta \cap \gamma$

$\exists \gamma \in P.$ ② $y \in \gamma \wedge z \in \gamma$

$$\Downarrow$$

$$\beta = \gamma$$

So $x \in \beta = \gamma \wedge z \in \gamma = \beta$.

$\boxtimes$

$$\begin{tikzcd} \underline{\mathrm{EqRel}}(A) \arrow[r, bend left, "\underline{\mathrm{part}}"] \arrow[r, phantom, "\cong"] & \underline{\mathrm{Part}}(A) \arrow[l, bend left, "\underline{\mathrm{eq}}"] \end{tikzcd}$$

iff

(1) $\forall E \in \underline{\mathrm{EqRel}}(A).$

$$\underline{\mathrm{eq}}(\underline{\mathrm{part}}(E)) = E$$

(2) $\forall P \in \underline{\mathrm{Part}}(A).$

$$\underline{\mathrm{part}}(\underline{\mathrm{eq}}(P)) = P$$

(1) Let $E \in \underline{EqRel}(A)$

$$\underline{eq}(\underline{part}(E))$$

$$= \{(x,y) \in A \times A \mid \exists \beta \in \underline{part}(E).\ x,y \in \beta\}$$

$$= \{(x,y) \in A \times A \mid \exists a \in A.\ x,y \in [a]_E\}$$

$$= \{(x,y) \in A \times A \mid x E y\}$$

$$= E$$

(2) Let $P \in \underline{\text{Part}}(A)$

Consider

$\underline{\text{part}}(\underline{\text{eq}}(P))$

$= \{ \alpha \subseteq A \mid \exists a \in A . \alpha = [a]_{\underline{\text{eq}}(P)} \}$

Since P is a partition, for every $a \in A$, there exists a unique $B(a) \in P$ such that $a \in B(a)$

Then, $[a]_{\underline{\text{eq}}(P)} = \{ x \in A \mid \exists \beta \in P . x \in \beta \wedge a \in \beta \}$

$= \{ x \in A \mid x \in B(a) \} = B(a)$

Hence

$$\underline{\text{part}}(\underline{\text{eq}}(P)) = \{\alpha \subseteq A \mid \exists a \in A.\ \alpha = B(a)\}$$

Moreover

$$\alpha \in P \iff \exists a \in A.\ \alpha = B(a)$$

Therefore

$$\underline{\text{part}}(\underline{\text{eq}}(P)) = \{\alpha \subseteq A \mid \alpha \in P\} = P$$

$\boxtimes$

Notation $E \subseteq A \times A$ equiv. rel.

notation

$$\text{part}(E) = A/_E = \{ [a]_E \mid a \in A \}$$

$$[a]_E = \{ x \in A \mid x \, E \, a \}.$$

$$\sim \subseteq (\mathbb{Z} \times \mathbb{N}^+) \times (\mathbb{Z} \times \mathbb{N}^+)$$

$(m,i) \circ \circ \bigcirc$ m/i

$(m,i) \sim (n,j)$ $\circ \circ \bigcirc$ $m/i = n/j$

iffdef $m \cdot j = n \cdot i$.

$$\mathbb{Q} = (\mathbb{Z} \times \mathbb{N}^+)/_\sim$$

Notation $f : A \cong B : g \iff f : A \to B,\ g : B \to A$
$f \circ g = id_B \wedge g \circ f = id_A$.
$(g = f^{-1} \wedge f = g^{-1})$.

Calculus of bijections

- $id_A : A \cong A : id_A$, $f : A \cong B : g \implies g : B \cong A : f$, $(f : A \cong B : g \wedge p : B \cong C : q) \implies p \circ f : A \cong C : g \circ q$

- If $A \cong X$ and $B \cong Y$ then

$$\mathcal{P}(A) \cong \mathcal{P}(X) \quad , \quad A \times B \cong X \times Y \quad , \quad A \uplus B \cong X \uplus Y \quad ,$$

$$\mathrm{Rel}(A, B) \cong \mathrm{Rel}(X, Y) \quad , \quad (A \rightharpoonup B) \cong (X \rightharpoonup Y) \quad ,$$

$$(A \Rightarrow B) \cong (X \Rightarrow Y) \quad , \quad \mathrm{Bij}(A, B) \cong \mathrm{Bij}(X, Y)$$