Inductive Definitions

The function

$$r : \mathbb{N} \to A$$

inductively defined from

$$a \in A$$

$$f : \mathbb{N} \times A \to A$$

is the unique such that

$$\begin{cases} r(0) = a \\ r(n+1) = f(n, r(n)) \quad n \in \mathbb{N} \end{cases}$$

Let A be a set. For $a \in A$ and a function $f : \mathbb{N} \times A \to A$,

Define

$$\mathcal{C} =^{def} \{ R \subseteq \mathbb{N} \times A \mid R \text{ is } (a,f)\text{-closed} \}$$

<u>Def</u>: R is (a,f)-closed

iff $\quad 0 \, R \, a$

and

$$\forall n \in \mathbb{N}, \forall a \in A . \; n \, R \, a \Rightarrow (n+1) \, R \, f(n,a)$$

Theorem

① The relation

$$r =_{\mathrm{def}} \bigcap \mathcal{G} : \mathbb{N} \nrightarrow A$$

is functional and total

② The function $r : \mathbb{N} \to A$ is the unique such that

$$r(0) = a$$

and

$$r(n+1) = f(n, r(n)) \text{ for all } n \in \mathbb{N}.$$

Lemma: r is (a,f)-closed.

Corollary: r is total.

$$\forall n \in \mathbb{N}.\ \exists x \in A.\ n\, r\, x.$$

Proof: By induction.

Base case $(n=0)$. RTP: $\exists x \in A.\ 0\, r\, x$

Indeed $0\, r\, a$

Inductive step For $n \in \mathbb{N}$.

(IH) $\exists x \in A.\ n\, r\, x$

RTP: $\exists y \in A.\ (n+1)\, r\, y$

Indeed if $n\,r\,x$ then $(n+1)\,r\,f(n,x)$ since r is (a,f)-closed. $\boxtimes$

Proposition r is functional.
There is only one pair (n,x) in r for all n.

PROOF: By induction.
Base case $(n=0)$. We know $(n,a)\in r$
Consider $r'\subseteq \mathbb{N}\times A$ defined as.
Def:
$$(0,a)\in r'$$
$$(n,x)\in r' \quad \forall n\geq 1,\ \forall x$$

- $r\subseteq r'$ which is (a,f)-closed

r' is functional at 0 and then so is r

Inductive step:

(IH) Suppose that r is functional at n

Consider $r' \subseteq \mathbb{N} \times A$ defined as

Def
$$i \, r' \, x \iff i \, r \, x \qquad \forall \, 0 \leq i \leq n$$
$$(n+1) \, r' \, f(n,y) \qquad \forall \, n \, r \, y$$
$$j \, r \, y \qquad \forall y \qquad \forall j > n+1$$

$r \subseteq r'$ r' is (a,f)-closed and functional at $n+1$ because by (IH) r is functional at n. Therefore r is functional at $n+1$

$\boxtimes$

Theorem 126 *The identity partial function is a function, and the composition of functions yields a function.*

NB

1. $f = g : A \to B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets A, the identity function $\mathrm{id}_A : A \to A$ is given by the rule

   $$\mathrm{id}_A(a) = a$$

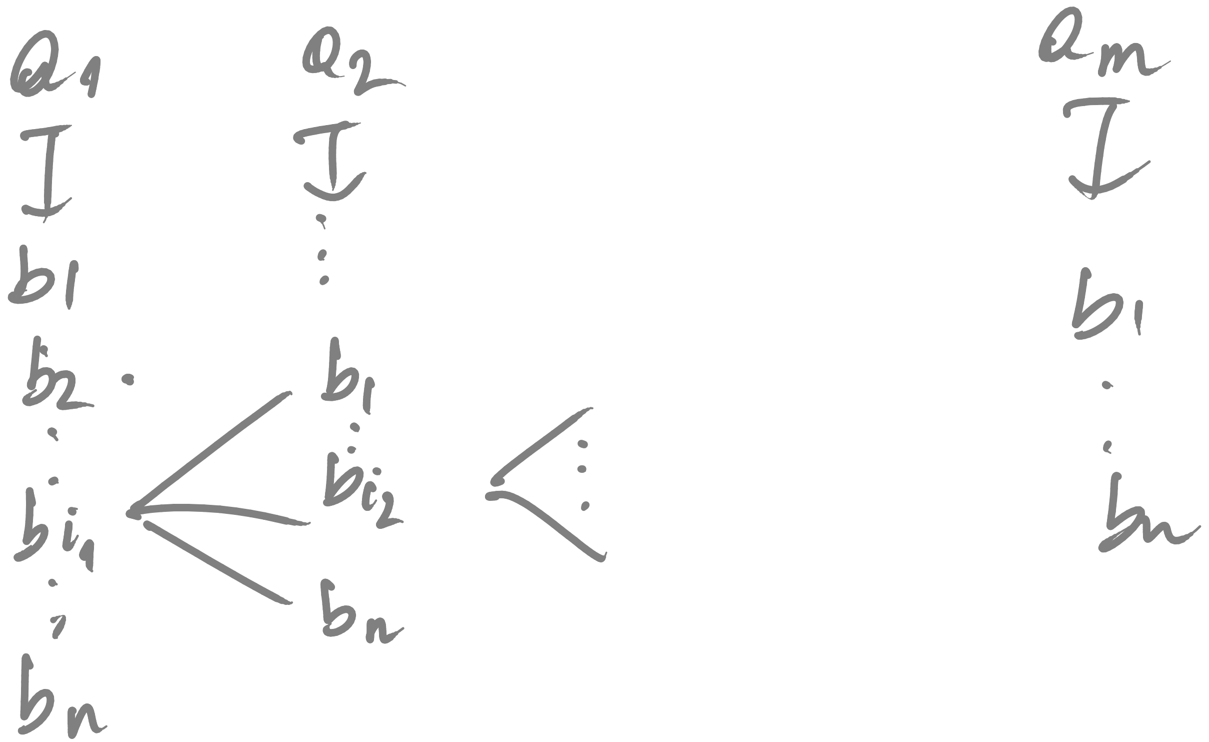
   and, for all functions $f : A \to B$ and $g : B \to C$, the composition function $g \circ f : A \to C$ is given by the rule

   $$\big(g \circ f\big)(a) = g\big(f(a)\big) \quad .$$

Proposition 125 *For all finite sets* A *and* B*,*

$$\#(A \Rightarrow B) = \#B^{\#A} \quad .$$

PROOF IDEA: $A = \{a_1 \ldots a_m\}$ $B = \{b_1 \ldots b_n\}$



$n \times n \times \ldots \times n = n^m$

Bijections

or invertible / reversible functions.

$$A \xrightarrow{f} B$$

is bijective

Def

$\exists g : B \to A .\ g \circ f = id_A$

$\wedge$

$\exists h : B \to A .\ f \circ h = id_B$

NB: If f is bijective and $g \circ f = id_A$ and $f \circ h = id_B$ then $h = g$

$$h = id \circ h = g \circ f \circ h = g \circ id = g$$

NB: Inverses of bijections are unique
The inverse of f is denoted f^{-1}.

Bijections

Definition 127 *A function* $f : A \to B$ *is said to be* bijective*, or a* bijection*, whenever there exists a (necessarily unique) function* $g : B \to A$ *(referred to as the* inverse *of* f*) such that*

1. g *is a* retraction *(or* left inverse*) for* f*:*

$$g \circ f = \mathrm{id}_A \quad ,$$

2. g *is a* section *(or* right inverse*) for* f*:*

$$f \circ g = \mathrm{id}_B \quad .$$

Examples

- $$\begin{tikzcd} \mathrm{Rel}([n],[n]) \arrow[r, bend left, "\mathrm{mat}"] & \text{Boolean } (n\times n)\text{-matrices} \arrow[l, bend left, "\mathrm{rel}"] \end{tikzcd}$$

- $$\begin{tikzcd} \mathbb{Z} \arrow[r, bend left, "\mathrm{succ}"] & \mathbb{Z} \arrow[l, bend left, "\mathrm{pred}"] \end{tikzcd}$$

  $\mathrm{succ}(n) = n+1$

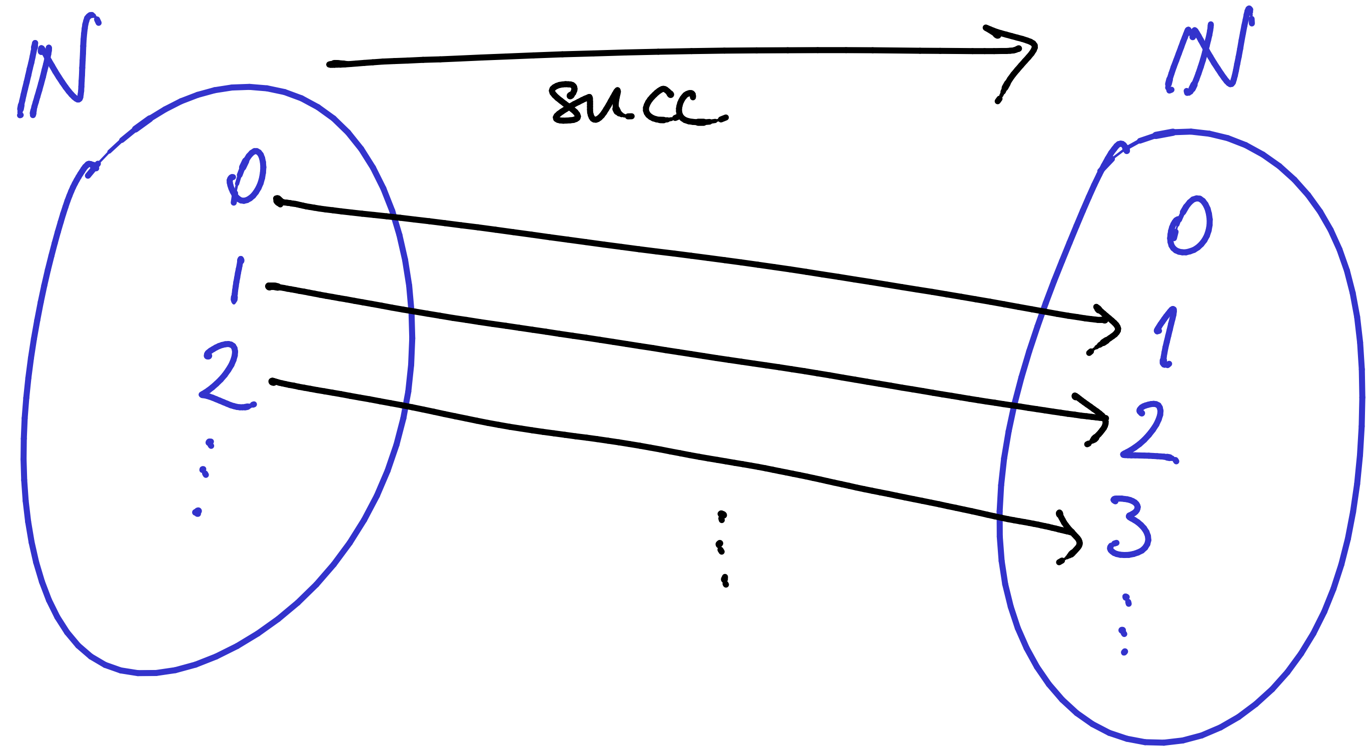
  $\mathrm{pred}(n) = n-1$

Non-example

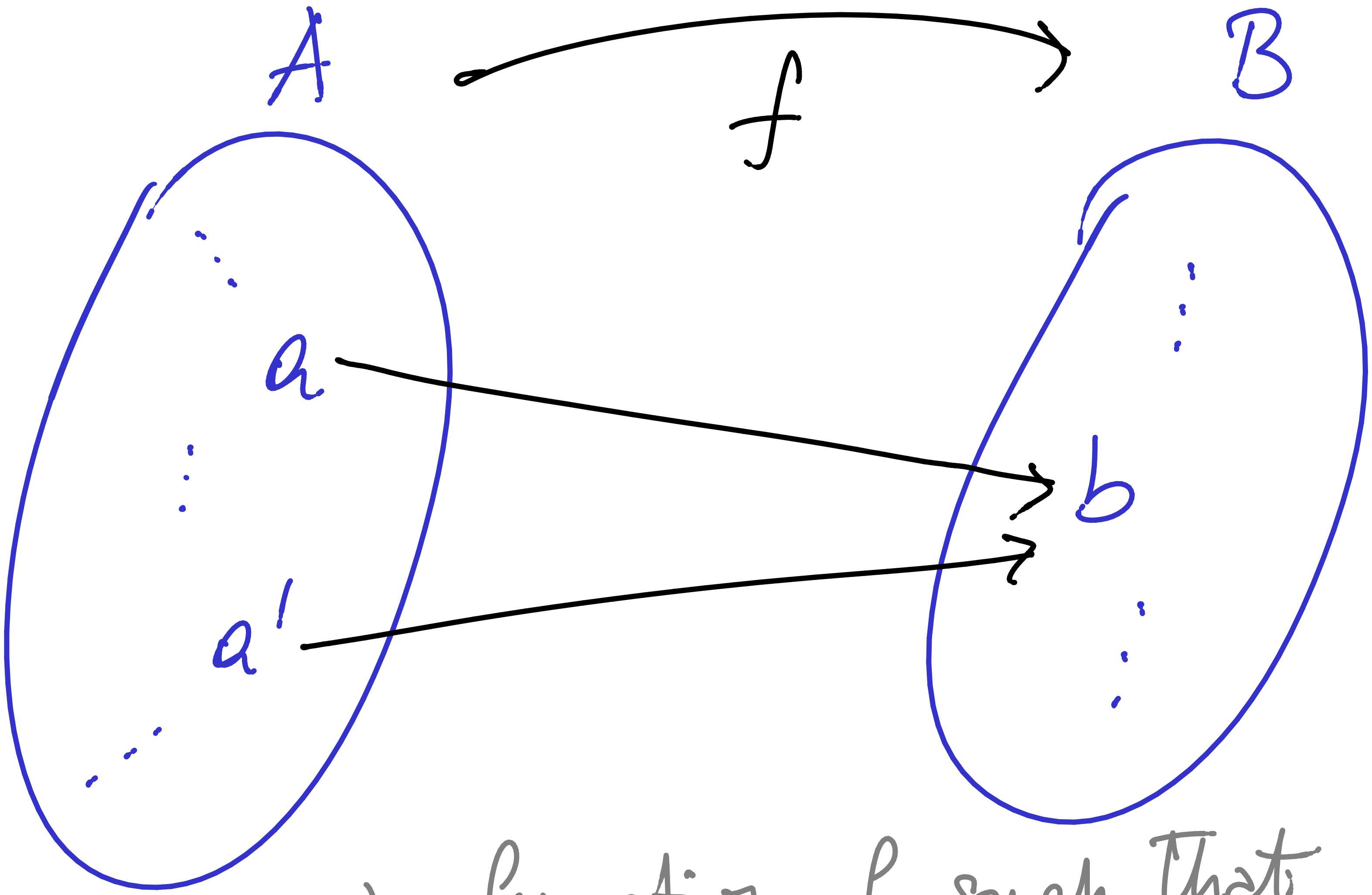
$$\begin{tikzcd} \mathbb{N} \arrow[r, "succ"] & \mathbb{N} \arrow[l, bend left, "g"] \end{tikzcd} \quad \text{is not a bijection}$$

$$\begin{tikzcd} & n \in \mathbb{N} \arrow[dl, mapsto] \arrow[d, phantom, "\|?"] \\ g(n) \arrow[r, mapsto, "\underline{succ}"'] & g(n)+1 \end{tikzcd}$$

$$\Longrightarrow \quad g(0)+1 = 0 \;\lightning$$



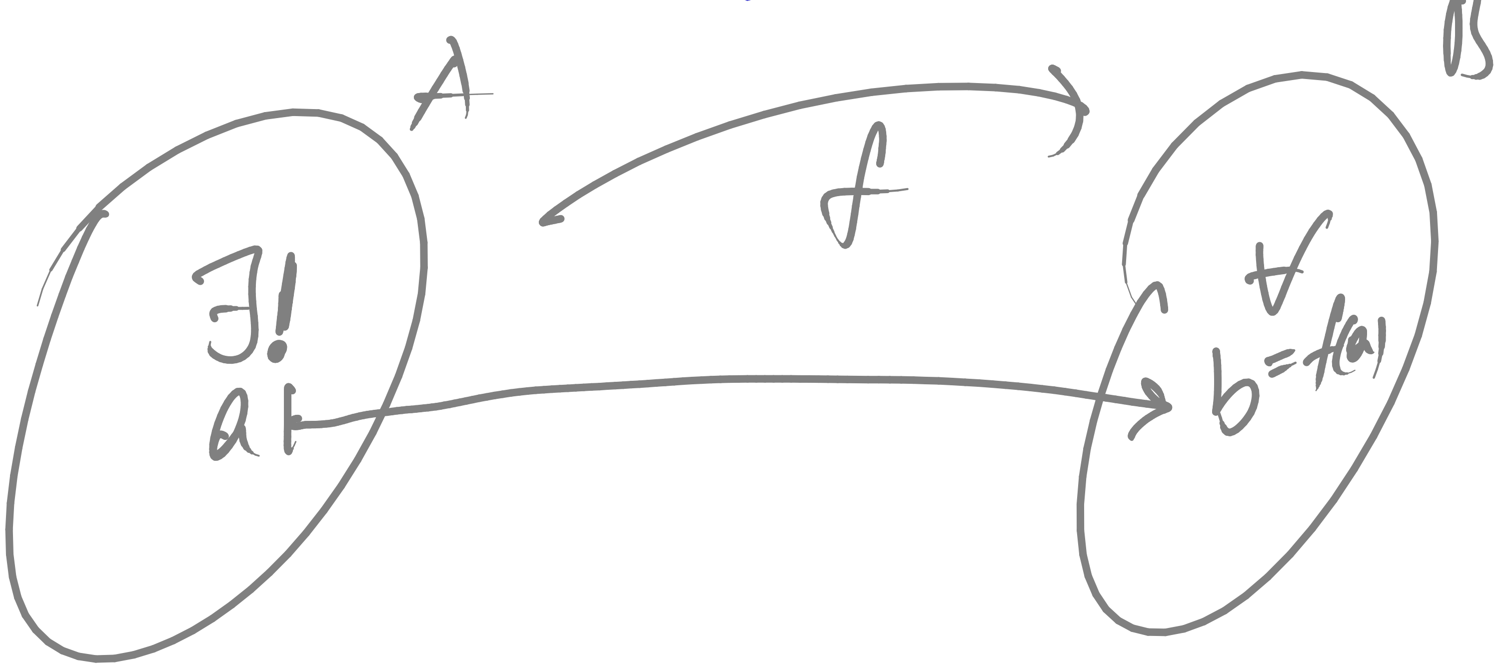
There is $k \in \mathbb{N}$, namely $k = 0$, such that, for all $n \in \mathbb{N}$, $\underline{\mathrm{succ}}(n) \neq k$.



a function f such that $f(a) = f(a')$ for $a \neq a'$ is not a bijection.

Proposition: A function $f: A \to B$ is a bijection if, and only if,

$$\forall b \in B.\ \exists!\, a \in A.\ f(a) = b.$$



Proposition 129 *For all finite sets* A *and* B*,*

$$\#\,\mathrm{Bij}(A,B) = \begin{cases} 0 & \text{, if } \#A \neq \#B \\ n! & \text{, if } \#A = \#B = n \end{cases}$$

PROOF IDEA:

$A = \{a_1, \ldots, a_m\}$ $\qquad$ $B = \{b_1, \ldots b_n\}$

If $m<n$ Then there is no bijection.

If $n<m$ Then there is no bijection.

If $n=m$ Then:

$$\begin{array}{lll}
a_1 \mapsto b_{i_1} & m \text{ choices.} & m! = m \\
a_2 \mapsto b_{i_2} & (m-1) \text{ choices} & \times (m-1) \\
a_3 \mapsto b_{i_3} & (m-2) \text{ choices} & \times (m-2) \\
\vdots & \vdots & \times \vdots \\
a_m \mapsto b_{i_m} & 1 \text{ choice} & \times 1 \quad \boxtimes
\end{array}$$

Theorem 130 *The identity function is a bijection, and the composition of bijections yields a bijection.*

$$\begin{tikzcd} A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & C \arrow[l, bend left, "g^{-1}"] \end{tikzcd}$$

NB: $(\mathrm{id}_A)^{-1} = \mathrm{id}_A$

For $f : A \to B$ and $g : B \to C$ bijections,
$g \circ f : A \to C$ bijection with inverse

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1} : C \to A .$$

<u>NB</u>: $\left(\underline{\mathrm{Bij}}(A, A), \mathrm{id}_A, \circ \right)$ is a group.

Definition 131 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \textit{or} \quad \#A = \#B \quad .$$

Examples:

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

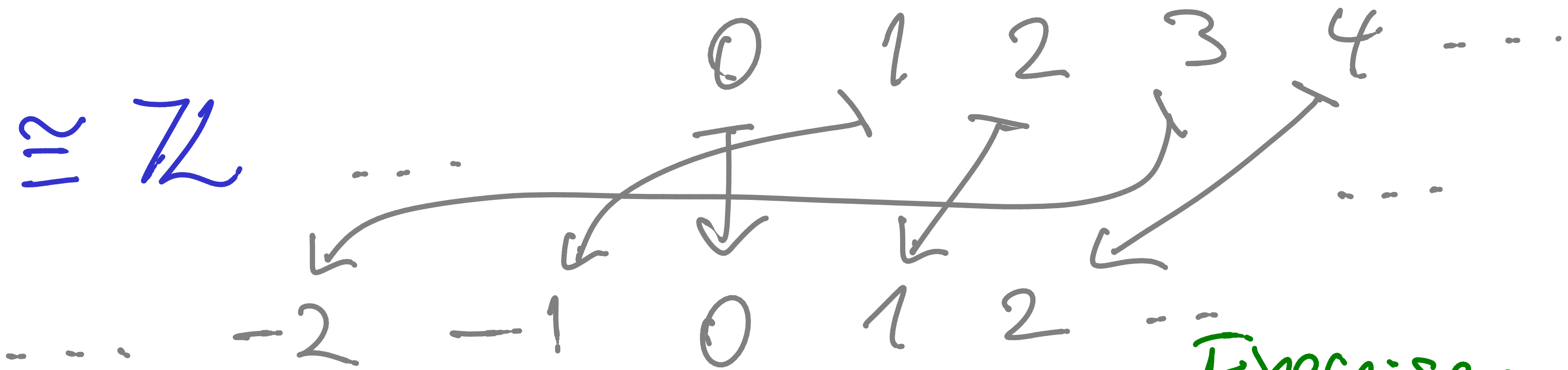
2. $\mathbb{N} \cong \mathbb{N}^+$, $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \cong \mathbb{Q}$.

Examples

$$\mathbb{N} \cong \mathbb{N}^+$$

$$\begin{tikzcd} \mathbb{N} \arrow[r, bend left, "succ"] & \mathbb{N}^+ \arrow[l, bend left, "pred"] \end{tikzcd}$$

$$\mathbb{N} \cong \mathbb{Z}$$



$$\begin{tikzcd} \mathbb{N} \arrow[r, bend left, "f"] & \mathbb{Z} \arrow[l, bend left, "g"] \end{tikzcd}$$

$$f(n) = \begin{cases} k & , \text{if } n = 2k \\ -(k+1), & \text{if } n = 2k+1 \end{cases}$$

Exercise: Define g such that $f \circ g = id_{\mathbb{Z}}$ and $g \circ f = id_{\mathbb{N}}$