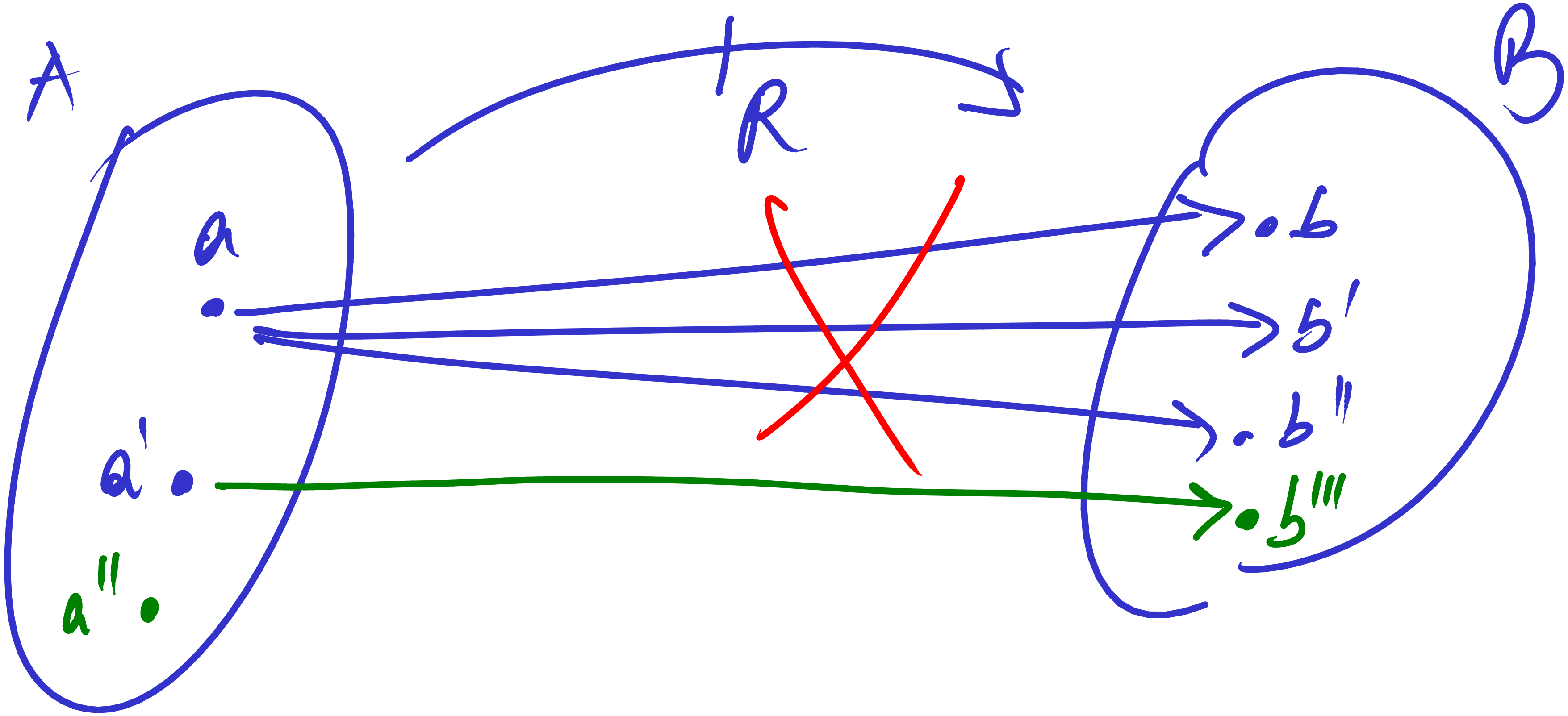
Partial functions

Definition 119 *A relation* $R : A \rightharpoonup\!\!\!\!\!\!\!+\, B$ *is said to be* functional, *and called a* partial function, *whenever it is such that*

$$\forall a \in A.\, \forall\, b_1, b_2 \in B.\ a\, R\, b_1 \,\wedge\, a\, R\, b_2 \implies b_1 = b_2 \quad .$$



Def $R : A \nrightarrow B$

R is <u>functional at</u> $a \in A$

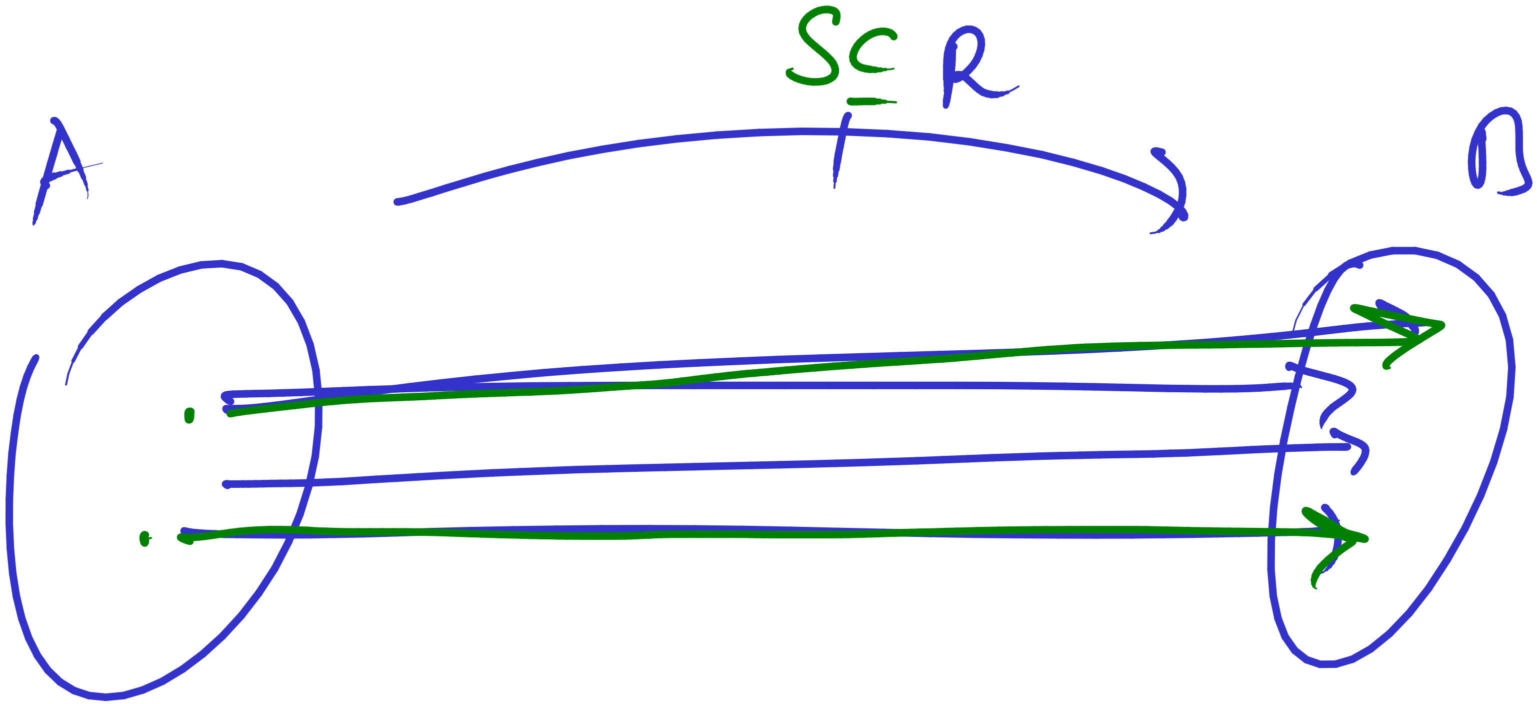
whenever a is related to at most one element of B

$$\forall b_1, b_2 \in B \quad a R b_1 \wedge a R b_2 \Rightarrow b_1 = b_2$$

NB: $S \subseteq R : A \nrightarrow B$

If R is functional at a

Then so is S.


A
$S \subseteq R$
B

Notation:

$f : A \to B$ $\quad$ f is a partial function from A to B

Given $a \in A$, we have either (i) There is no $b \in B$ such that $a f b$

or (ii) There is a unique $b \in B$ such that $a f b$

In case (i), we write

$f(a)\uparrow$ $\quad$ f is undefined at a

In case (ii), we write

$f(a)\downarrow$ $\quad$ f is defined at a

Moreover,

$f(a)$ denotes the unique element of B such that $(a, f(a))$ is in f.

Domain of definition

For $f : A \to B$,

$$\underline{\mathrm{dom}}(f) \subseteq A$$

$$\|\mathrm{def}$$

$$\{ a \in A \mid f(a)\downarrow \} = \{ a \in A \mid \exists b \in B. \; a f b \}.$$

Example:

$$\underline{pred} : \mathbb{N} \to \mathbb{N}$$

$$\overset{\underline{def}}{=} \{(n+1, n) \in \mathbb{N} \times \mathbb{N} \mid n \in \mathbb{N}\}$$

$$= \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \geq 1 \wedge x = y + 1\}$$

- $\underline{pred}(0) \uparrow$
- $\underline{pred}(m) \downarrow$ for m a positive integer

  $\underline{pred}(m) = m - 1$
- $\underline{dom}(\underline{pred})$ is the set of positive integers

Defining partial functions

$$f : A \to B$$

$$f : \underset{\in A}{a} \longmapsto \underbrace{\cdots\cdots a \cdots\cdots}_{\in B}$$

rule,
mapping,
assignment,
definition,
construction,
etc.

Example:

$$\underline{pred} : \mathbb{N} \longrightarrow \mathbb{N}$$

$$\underline{pred} : \underset{\mathbb{N}}{\underset{\cap}{n}} \longmapsto \max_{k \in \mathbb{N}} \; k < n$$

as a relation:

$$\underline{pred} = \{ (n,m) \in \mathbb{N} \times \mathbb{N} \mid m = \max_{k \in \mathbb{N}} \; k < n \}$$

For all $n \in \mathbb{N}$ there is at most one element equal to $\max_{k \in \mathbb{N}} k < n$. For $n = 0$ there is no such element, for $n \geq 1$ that element is $n-1$.

Example: Quotient with remainder for integers

$$qr : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{N}$$

$$\underline{\text{dom}}\,(qr) = \{ (n,m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0 \}$$

$$qr : (n,m) \mapsto (q,r) \in \mathbb{Z} \times \mathbb{N}$$

such that $n = q \cdot m + r$

with $0 \leq r < m$

Example: The following defines a partial function $\mathbb{Z} \times \mathbb{Z} \rightharpoonup \mathbb{Z} \times \mathbb{N}$:

- for $n \geq 0$ and $m > 0$,
  $$(n, m) \mapsto \big(\operatorname{quo}(n, m)\,,\ \operatorname{rem}(n, m)\big)$$
- for $n \geq 0$ and $m < 0$,
  $$(n, m) \mapsto \big(-\operatorname{quo}(n, -m)\,,\ \operatorname{rem}(n, -m)\big)$$
- for $n < 0$ and $m > 0$,
  $$(n, m) \mapsto \big(-\operatorname{quo}(-n, m) - 1\,,\ \operatorname{rem}(m - \operatorname{rem}(-n, m), m)\big)$$
- for $n < 0$ and $m < 0$,
  $$(n, m) \mapsto \big(\operatorname{quo}(-n, -m) + 1\,,\ \operatorname{rem}(-m - \operatorname{rem}(-n, -m), -m)\big)$$

Its domain of definition is $\{\,(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0\,\}$.

Notation:

The set of all relations from A to B

$$(A \rightharpoonup B) \subseteq \underline{Rel}(A,B) = \mathcal{P}(A \times B)$$

The set of all partial functions from A to B

- $f = g : A \rightharpoonup B$

Iff $\forall a \in A.\ \big(f(a)\downarrow \iff g(a)\downarrow\big)$

$\wedge \big[f(a)\downarrow \wedge\ g(a)\downarrow \Rightarrow f(a) = g(a) \big]$

Identities and Composition

- $\mathrm{id}_A \in \underline{\mathrm{Rel}}(A,A)$
  is a partial function $A \to A$

- Let $f: A \to B$ and $g: B \to C$.
  Consider $g \circ f \in \underline{\mathrm{Rel}}(A,C)$

  $$g \circ f \overset{\text{def}}{=} \{(a,c) \in A \times C \mid \exists b \in B.\, a f b \wedge b g c\}$$

Theorem 121 *The identity relation is a partial function, and the composition of partial functions yields a partial function.*

NB

$$f = g : A \rightharpoonup B$$

iff

$$\forall a \in A. \big(f(a)\downarrow \iff g(a)\downarrow \big) \wedge f(a) = g(a)$$

$f : A \rightharpoonup B$, $g : B \rightharpoonup C$ $\leadsto$ $g \circ f : A \rightharpoonup C$

$$(g \circ f)(a) = \begin{cases} \uparrow & , \text{ if } f(a)\uparrow \\ \uparrow & , \text{ if } f(a)\downarrow \text{ but } g(f(a))\uparrow \\ g(f(a)) & , \text{ if } f(a)\downarrow \text{ and } g(f(a))\downarrow \end{cases}$$

Proposition 122 *For all finite sets* A *and* B,

$$\#(A \rightharpoonup B) \;=\; (\#B + 1)^{\#A} \quad .$$

PROOF IDEA: $A = \{a_1, \ldots, a_m\}$

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_m \\ \downarrow & \downarrow & & \downarrow \\ b_1 & b_1 & & b_1 \\ \vdots & \vdots & & \vdots \\ b_n & b_n & & b_n \\ \uparrow & \uparrow & & \uparrow \end{array}$$

in $B = \{b_1, \ldots, b_n\}$

$$\underbrace{(m+1) \times \cdots \times (m+1)}_{n \text{ times}}$$

$\boxtimes$

Functions

$$(A \Rightarrow B) \subseteq (A \rightharpoonup B) \subseteq \underline{\mathrm{Rel}}(A, B)$$

The set of all functions from A to B

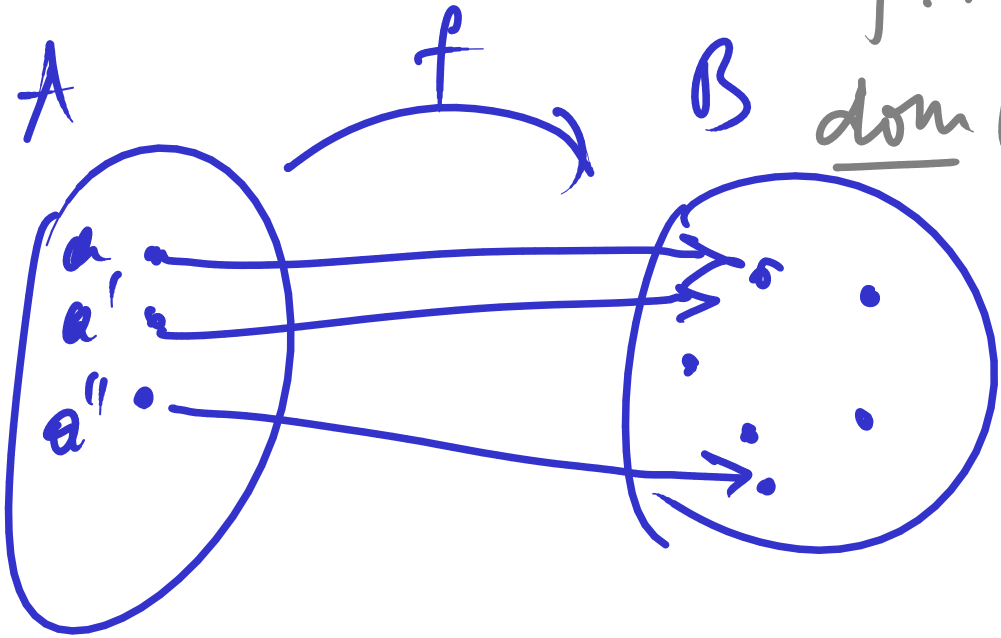
Functions (or maps)

Definition 123 *A partial function is said to be total, and referred to as a (total) function or map, whenever its domain of definition coincides with its source.*

$f : A \to B$

$\underline{\mathrm{dom}}(f) = A \rightsquigarrow f$ total

$f : A \to B$. $\underline{\text{notation}}$



Theorem 124 *For all* $f \in \mathrm{Rel}(A, B)$,

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\, \exists!\, b \in B.\ a\, f\, b \quad .$$

Example: Total predecessor function.

$\underline{\text{totpred}} : \mathbb{N} \to \mathbb{N}$

$$\underline{\text{totpred}}(n) = \begin{cases} 0 & \text{if } n = 0 \\ n-1 & \text{if } n \geq 1 \end{cases}$$

Inductive Definitions

Example:

$\underline{add} : \mathbb{N}^2 \to \mathbb{N}$

$$\begin{cases} \underline{add}(m, 0) \stackrel{def}{=} m \\ \underline{add}(m, n+1) \stackrel{def}{=} \underline{add}(m, n) + 1 \end{cases}$$

Example: $t : \mathbb{N} \to \mathbb{N}$

$$t(n) = \sum_{i=0}^{n} i \, .$$

$$\begin{cases} t(0) = 0 \\ t(n+1) = \underline{add}(n, t(n)) \end{cases}$$

Inductive Definitions

The function

$$r : \mathbb{N} \to A$$

inductively defined from

$$a \in A$$

$$f : \mathbb{N} \times A \to A$$

is the unique such that

$$\begin{cases} r(0) = a \\ r(n+1) = f(n, r(n)) \quad n \in \mathbb{N} \end{cases}$$

NB:

for fixed $m \in \mathbb{N}$.

$$\underline{add}_m : \mathbb{N} \to \mathbb{N}$$

$$\begin{cases} \underline{add}_m(0) = m \\ \underline{add}_m(n+1) = \underline{add}_m(n) + 1 \end{cases}$$

$$\underline{add} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\underline{add}(m, n) = \underline{add}_m(n)$$

Let A be a set. For $a \in A$ and a function $f: \mathbb{N} \times A \to A$,

Define

$$\mathcal{C} =^{def} \{ R \subseteq \mathbb{N} \times A \mid R \text{ is } (a,f)\text{-closed} \}$$

Def: R is (a, f)-closed

iff $\quad 0 \, R \, a$

and

$\forall n \in \mathbb{N}, \forall a' \in A.\ n \, R \, a' \Rightarrow (n+1) \, R \, f(n, a')$

$r =^{def} \bigcap \mathcal{C}$ the set of all (a,f)-closed relations.

$\underbrace{r}_{}$ inductively defined by (a,f) is the least (a,f)-closed relation.

Thm: $\underset{\mathbb{N} \nrightarrow A}{r}$ is total functional relation.

total: $\forall n \in \mathbb{N}. \exists a_n \in A.\ n\, r\, a_n$

functional: $\forall n \in \mathbb{N}.\ n\, r\, x \wedge n\, r\, y \Rightarrow x = y.\ \forall x, y \in A.$

<u>Lemma</u>: $r = \bigcap \mathcal{C}$ is (a,f)-closed.

$i\ r\ x \iff \forall\ (a,f)$-closed R. $i\ R\ x$

$0\ r\ a$?

$\Updownarrow$

$\forall\ (a,f)$-closed R, $0\ R\ a$ which is the case ✓

$n\ r\ x \overset{?}{\Longrightarrow} (n+1)\ r\ f(n,x)$

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$\forall (a,f)$-closed R $\quad\Longrightarrow\quad$ $\forall (a,f)$-closed R.

$n\ R\ x$ $\qquad\qquad\qquad\qquad$ $(n+1)\ R\ f(n,x)$

Theorem

① The relation

$$r =_{def} \bigcap \mathcal{C} : \mathbb{N} \nrightarrow A$$

is functional and total

② The function $r : \mathbb{N} \to A$ is the unique such that

$$r(0) = a$$

and

$$r(n+1) = f(n, r(n)) \text{ for all } n \in \mathbb{N}.$$