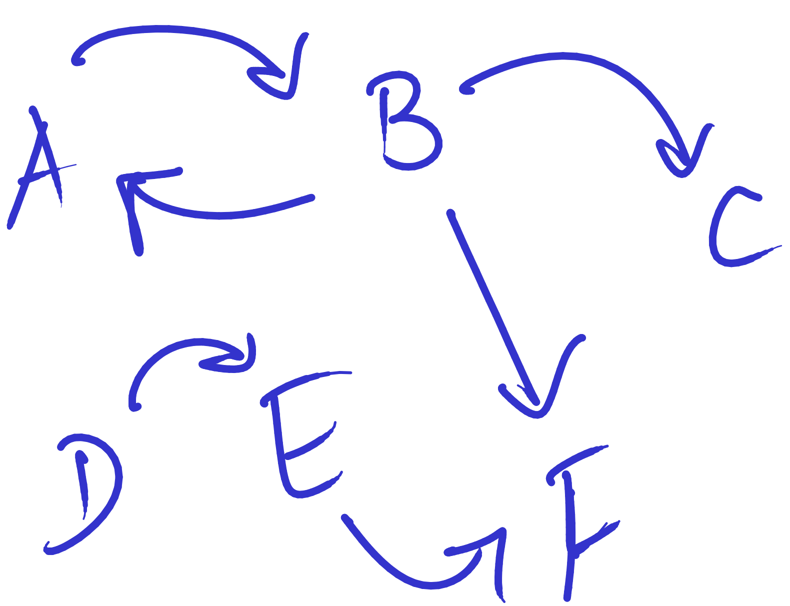
Directed graphs

Definition 108 *A* directed graph (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*



$R \subseteq A \times A$

<u>Idea</u>: there is a connection from s to t iff $(s,t) \in R$.

$(\underline{Rel}(A), \mathrm{id}_A, \circ)$ is a monoid.

$R \in \underline{Rel}(A)$

$\underbrace{R}_{R^{\circ(1)}}, \underbrace{R \circ R}_{R^{\circ(2)}}, \underbrace{R \circ R \circ R}_{R^{\circ(3)}}, \ldots, \underbrace{R \circ \cdots \circ R}_{n \text{ times}}, \ldots$

$R^{\circ(n)}$

$R^{\circ(0)} \stackrel{\text{def}}{=} \mathrm{id}_A$

$R^{\circ(n+1)} = R^{\circ(n)} \circ R$

$= R \circ R^{\circ(n)}$

$x \, R^{\circ(2)} \, y$

$\iff \exists z. \; xRz \wedge zRy$

$x \, R^{\circ(3)} \, y$

$\iff \exists z. \; x R^{\circ(2)} z \wedge zRy \iff \exists z, u. \; xRu \wedge uRz \wedge zRy.$

Corollary 110 *For every set A, the structure*

$$\left(\ \mathrm{Rel}(A)\ ,\ \mathrm{id}_A\ ,\circ\ \right)$$

is a monoid.

Definition 111 *For $R \in \mathrm{Rel}(A)$ and $n \in \mathbb{N}$, we let*

$$R^{\circ n}\ =\ \underbrace{R \circ \cdots \circ R}_{n \textit{ times}}\ \in \mathrm{Rel}(A)$$

be defined as id_A for $n = 0$, and as $R \circ R^{\circ m}$ for $n = m + 1$.

Paths

Proposition 113 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s \; R^{\circ n} \; t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

PROOF:

Paths

A path of length n from s to t is a sequence

$$s = a_0 R a_1 R a_2 \ldots R a_n = t$$

NB: There is always a path of length 0 from a node to itself.

PROOF $s \, R^{o(n)} \, t$

$\iff \exists$ path of length n from s to t.

By induction on $n \in \mathbb{N}$.

BASE CASE $(n = 0)$:

$$
\begin{array}{ccc}
s \, R^{o(0)} \, t & \overset{?}{\iff} & \exists \text{ path of length } 0 \text{ from } s \text{ to } t \\
\Updownarrow & & \Updownarrow \\
s \, \mathrm{id}_A \, t & \iff & s = t
\end{array}
$$

INDUCTIVE STEP

(IH) $s \, R^{(n)} \, t \iff \exists$ path of length n from s to t.

RTP: ? $s \, R^{(n+1)} \, t$

$\iff \exists$ path of length $n+1$ from s to t

($\Rightarrow$)

$s \, R^{(n+1)} \, t \iff s \, R^{(n)} \, z \wedge z \, R \, t$ for some z

By (IH): $\exists$ path of length n from s to z, say

$s = a_0 \, R \, a_1 \, R \cdots R \, a_n = z$

So $s = a_0 \, R \, a_1 \, R \cdots R \, a_n \, R \, a_{n+1} = t$ is a path of length $n+1$ from s to t.

($\Leftarrow$) RTP: $\exists$ path of length $n+1$ from s to t

$\Rightarrow$ $s R^{\circ(n+1)} t$

Assume $s = a_0 R a_1 R \dots R a_n R a_{n+1} = t$

Then $s = a_0 R a_1 R \dots R a_n$ is a path of length n from s to a_n. So by (IH): $s R^{\circ(n)} a_n$. Moreover $a_n R t$. Therefore $s \underbrace{(R^{\circ(n)} \circ R)}_{R^{\circ(n+1)} \text{ by def.}} t$. $\boxtimes$

$x \, R^{\circ *} \, y \iff \exists n \in \mathbb{N}. \; x \, R^{\circ(n)} \, y$

$\iff \exists n \in \mathbb{N}.$ $\exists$ path of length n from x to y

$\iff \exists$ (finite) path from x to y.

Definition 114 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} = \bigcup \{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} R^{\circ n} \quad .$$

Corollary 115 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s \; R^{\circ *} \; t$ *iff there exists a path with source* s *and target* t *in* R.

NB Suppose $A = [n] = \{0, 1, \ldots, n-1\}$

$R^{\circ *} = \mathrm{Id}_A \cup R \cup R^{\circ(2)} \cup R^{\circ(3)} \cup \cdots \cup R^{\circ(n-1)}$.

$R \subseteq [n] \times [n] \quad \rightsquigarrow \quad R^{\circ *}$

$\mathrm{mat}(R) = M$ adjecency matrix of R

$$R^{\circ *} = \mathrm{id}_A \cup R \cup R^{\circ(2)} \cup \dots \cup R^{\circ(n-1)}$$

$$\underbrace{\mathrm{mat}(R^{\circ *})}_{\substack{\| \mathrm{def} \\ M^*}} = \underbrace{\mathrm{mat}(\mathrm{id}_{[n]})}_{\substack{\| \\ I_n}} + M + \underbrace{\mathrm{mat}(R^{\circ 2})}_{\substack{\| \\ \mathrm{mat}(R) \cdot \mathrm{mat}(R) \\ \| \\ M \cdot M \\ \| \\ M^2}} + \dots + \underbrace{\mathrm{mat}(R^{\circ(n-1)})}_{\substack{\| \\ M^{n-1}}}$$

$$M^* = I_n + M + M^2 + \dots + M^{n-1}$$

$M_0 = I_n$

$M_1 = I_n + (M \cdot M_0) = I_n + M \cdot I_n = I_n + M$

$M_2 = I_n + M \cdot M_1 = I_n + M(I_n + M) = I_n + M \cdot I_n + M^2$

$= I_n + M + M^2$

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its *adjacency matrix*.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 = I_n \\ M_{k+1} = I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

Preorders

Definition 116 *A* preorder $(P, \sqsubseteq)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

- *Reflexivity.*

$$\forall x \in P.\ x \sqsubseteq x$$

- *Transitivity.*

$$\forall x, y, z \in P.\ (x \sqsubseteq y \ \wedge \ y \sqsubseteq z) \implies x \sqsubseteq z$$

Partial order: A preorder such that (antisymmetry)

$$x \sqsubseteq y \wedge y \sqsubseteq x \implies x = y$$

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\mathbb{Z}, \mid)$.

note $n \mid -n$ and $-n \mid n$ but $n \neq -n$ for $n \neq 0$

Theorem 118 *For* $R \subseteq A \times A$, *let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \textit{ is a preorder} \} \quad .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $\boxed{R^{\circ *} = \bigcap \mathcal{F}_R.}$

PROOF:

$R^{\circ *}$ is the least preorder that contains R

$R^{\circ *} \in \mathcal{F}_R \Rightarrow \bigcap \mathcal{F}_R \subseteq R^{\circ *}$

(i) $R^{\circ *} \in \mathcal{F}_R$

$\iff R \subseteq R^{\circ *}$ and $R^{\circ *}$ is a preorder. exercise.

(ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$

$\iff \bigcup_{n \in \mathbb{N}} R^{\circ (n)} \subseteq \bigcap \mathcal{F}_R$

$\iff \forall n \in \mathbb{N}.\ R^{\circ (n)} \subseteq \bigcap \mathcal{F}_R$

$\iff \forall n \in \mathbb{N}.\ \forall Q \in \mathcal{F}_R.\ R^{\circ (n)} \subseteq Q.$

By induction on $n \in \mathbb{N}$.

$\bigcup \mathcal{F} \subseteq X$

iff.

$\forall A \in \mathcal{F}.$

$A \subseteq X$

Base case $(n=0)$ $\quad Id \subseteq Q \quad$ because Q is reflexive.

Ind. Step.

(IH) $\quad \forall Q \in \mathcal{F}_R.\ R^{\circ(n)} \subseteq Q.$

Lemma.

$$\frac{R_1 \subseteq S_1 \qquad R_2 \subseteq S_2}{R_1 \circ R_2 \subseteq S_1 \circ S_2}$$

RTP: $\quad \forall Q \in \mathcal{F}_R.\ R^{\circ(n+1)} \subseteq Q.$

Let $Q \in \mathcal{F}_R$.

Lemma

Q transitive $\iff Q \circ Q \subseteq Q$

$$R^{\circ(n+1)} = R^{\circ(n)} \circ R \underset{(IH)}{\subseteq} Q \circ Q \underbrace{\subseteq}_{\text{by transitivity}} Q$$

$\boxtimes$