Relations

Definition 99 *A (binary) relation R from a set A to a set B*

$$R : A \longrightarrow\!\!\!\!\!\!\!+\ B \quad \textit{or} \quad R \in \mathrm{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \quad \textit{or} \quad R \in \mathcal{P}(A \times B) \quad .$$

Notation 100 *One typically writes $a\,R\,b$ for $(a, b) \in R$.*

Informal examples:

- Computation.
- Typing.
- Program equivalence.
- Networks.
- Databases.

Examples:

- Empty relation.

  $\emptyset : A \longrightarrow\!\!\!\!\!+\; B$ $\qquad$ $(a\ \emptyset\ b \iff \textbf{false})$

- Full relation.

  $(A \times B) : A \longrightarrow\!\!\!\!\!+\; B$ $\qquad$ $(a\ (A \times B)\ b \iff \textbf{true})$

- Identity (or equality) relation.

  $\mathrm{id}_A = \{\ (a, a) \mid a \in A\ \} : A \longrightarrow\!\!\!\!\!+\; A$ $\qquad$ $(a\ \mathrm{id}_A\ a' \iff a = a')$

- Integer square root.

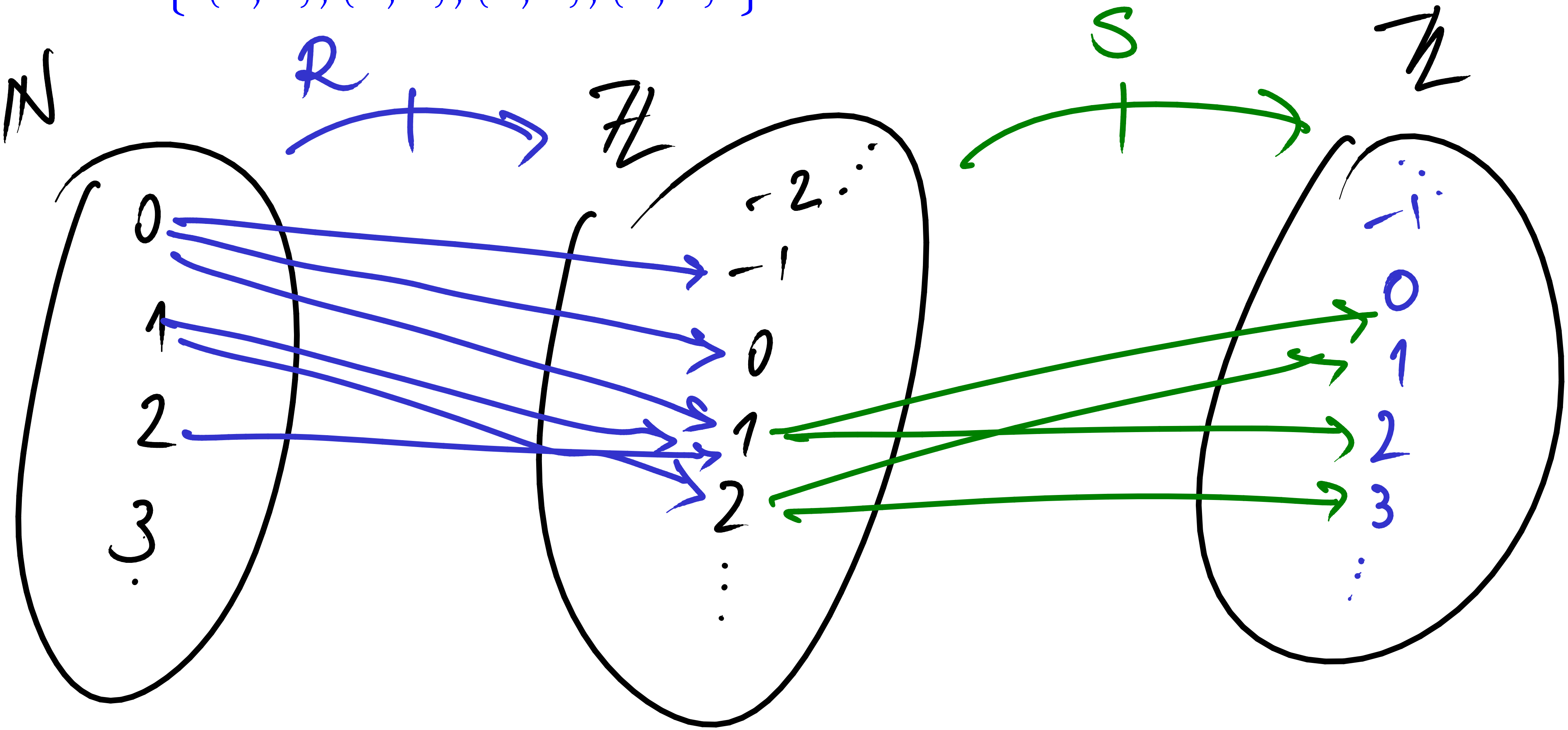
  $R_2 = \{\ (m, n) \mid m = n^2\ \} : \mathbb{N} \longrightarrow\!\!\!\!\!+\; \mathbb{Z}$ $\qquad$ $(m\ R_2\ n \iff m = n^2)$

Internal diagrams

Example:

$$R = \{\ (0,0), (0,-1), (0,1), (1,2), (1,1), (2,1)\ \} : \mathbb{N} \rightarrow\!\!\!\!\!\!+\ \mathbb{Z}$$

$$S = \{\ (1,0), (1,2), (2,1), (2,3)\ \} : \mathbb{Z} \rightarrow\!\!\!\!\!\!+\ \mathbb{Z}$$



Relational extensionality

$$R = S : A \twoheadrightarrow B$$

iff

$$\forall a \in A.\, \forall b \in B.\ a\,R\,b \iff a\,S\,b$$

Relational composition

$$\frac{A \xrightarrow{R} B \qquad B \xrightarrow{S} C}{A \xrightarrow[S \circ R]{} C}$$

notation:

$R;S$

$S \circ R \subseteq A \times C$

$\forall a \in A,\ c \in C.$

$$a(S \circ R)c \overset{\text{def.}}{\iff} \exists b \in B.\ aRb \wedge bSc$$

Example

$Sq : \mathbb{R}_{\geq 0} \not\rightarrow \mathbb{R}$ $\qquad$ $a \, Sq \, b \overset{def}{\iff} a = b^2$

$Neg : \mathbb{R} \not\rightarrow \mathbb{R}$ $\qquad$ $x \, Neg \, y \overset{def}{\iff} y = -x$

Claim:

$$Neg \circ Sq = Sq : \mathbb{R}_{\geq 0} \not\rightarrow \mathbb{R}$$

RTP: $\forall s \in \mathbb{R}_{\geq 0}, t \in \mathbb{R}.$

$$s \, (Neg \circ Sq) \, t \overset{?}{\iff} s \, Sq \, t \overset{\text{by def}}{\iff} s = t^2$$

$$\Big\Updownarrow \text{def} \qquad\qquad\qquad\qquad \Big\Updownarrow \text{exercise}$$

$$\exists r \in \mathbb{R}.\ s \, Sq \, r \wedge r \, Neg \, t \iff \exists r \in \mathbb{R}.\ s = r^2 \wedge t = -r$$

Theorem 102 *Relational composition is associative and has the identity relation as neutral element.*

NB. Unambiguously

$T \circ S \circ R$

- *Associativity.*

  For all $R : A \rightarrowtriangle B$, $S : B \rightarrowtriangle C$, *and* $T : C \rightarrowtriangle D$,

$$(T \circ S) \circ R = T \circ (S \circ R)$$

- *Neutral element.*

  For all $R : A \rightarrowtriangle B$,

$$R \circ \mathrm{id}_A = R = \mathrm{id}_B \circ R \quad .$$

$$(T \circ S) \circ R = T \circ (S \circ R) : A \nrightarrow D$$

$\forall a \in A, d \in D.$

$$a\,((T \circ S) \circ R)\,d \overset{?}{\iff} a\,(T \circ (S \circ R))\,d$$

$\Updownarrow$

$\exists b \in B.\ aRb \wedge b\,(T \circ S)\,d$

$\Updownarrow$

$\exists b \in B.\ aRb \wedge \exists c \in C.\ bSc \wedge cTd$

$\Updownarrow$

$\exists b \in B.\ \exists c \in C.\ aRb \wedge bSc \wedge cTd.$

$a\,(T \circ (S \circ R))\,d$

$\Updownarrow$

$\exists c \in C.\ a\,(S \circ R)\,c \wedge cTd$

$\Updownarrow$

$\exists c \in C.\ \exists b \in B.\ aRb \wedge bSc \wedge cTd$

Definition 103

1. *For positive integers* m *and* n*, an* $(m \times n)$*-*<u>matrix</u> M *over a* comm. *semiring* $(S, 0, \oplus, 1, \odot)$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \leq i < m$ *and* $0 \leq j < n$.

$$M = \begin{array}{cc} & \begin{array}{cccccc} 0 & 1 & \cdots & j & \cdots & n-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \\ \vdots \\ m-1 \end{array} & \left[\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & M_{i,j} & & \\ & & & & & \\ & & & & & \end{array} \right] \end{array}$$

Theorem 104 *Matrix multiplication is associative and has the identity matrix as neutral element.*

M $(m \times n)$-matrix

N $(n \times \ell)$-matrix

$N \bullet M$ $(m \times \ell)$-matrix

I $(m \times m)$-matrix

$$I_{i,j} = \begin{cases} 1 & i = j \\ 0 & \text{otw} \end{cases}$$

$$(N \bullet M)_{i,j} = \sum_{k} M_{i,k} \odot N_{k,j}$$

iterated $\oplus$ of S

Recall: $[k] = \{0, 1, 2, \ldots, k-1\}$ $k \in \mathbb{N}$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

Given

$R : [m] \nrightarrow [n]$

Define

$\underline{mat}(R)$ $(m \times n)$-matrix

$$(\underline{mat}(R))_{i,j} = \begin{cases} true & iRj \\ false & otw \end{cases}$$

$i \in [m], j \in [n]$

$Bool = \{true, false\}$

$\oplus = \vee$

$\odot = \wedge$

$$R : [3] \nrightarrow [2]$$

$$\underline{\mathrm{mat}}(R) = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \text{true} & \text{false} \\ \text{false} & \text{true} \\ \text{false} & \text{false} \end{bmatrix} \end{array}$$

$$R = \{ (0,0), (1,1) \}$$

Given $(m \times n)$-matrix M, define $\underline{rel}(M) : [m] \rightarrow [n]$

$\forall i \in [m], j \in [n],$

$$i \, (\underline{rel}(M)) \, j \overset{def}{\iff} (M_{i,j} = \text{true})$$

<u>Proposition</u> :

$$\underline{rel}\,(\underline{mat}(R)) = R$$

$$\underline{mat}(\underline{rel}(M)) = M$$

Proposition: $[m] \overset{R}{\nrightarrow} [n]$, $[n] \overset{S}{\nrightarrow} [\ell]$

$\mathrm{mat}(R)$ $(m \times n)$-matrix $\qquad$ $\underline{\mathrm{mat}}(S)$ $(n \times \ell)$-matrix

$$\underline{\mathrm{mat}}(S \circ R) = \underline{\mathrm{mat}}(S) \bullet \underline{\mathrm{mat}}(R) \qquad (*)$$

Corollary:

$$S \circ R = \underline{\mathrm{rel}}\left(\underline{\mathrm{mat}}(S) \bullet \underline{\mathrm{mat}}(R) \right)$$

$$\left(\underline{mat}(S \circ R)\right)_{i,j} = \underline{true}$$

$$\Leftrightarrow \quad i\,(S \circ R)\,j$$

$$\Leftrightarrow \quad \exists k .\ iRk \wedge kSj$$

$$\left(\underline{mat}(S) \bullet \underline{mat}(R)\right)_{i,j}$$

$$= \bigvee_k .\ \underline{mat}(R)_{i,k} \wedge \underline{mat}(S)_{k,j}$$

$$= \bigvee_k \ (i,k) \in R \wedge (k,j) \in S$$

$$\underline{mat}(id) = I$$

Def M, N $(m \times n)$ matrices.

$$(M+N)_{i,j} = M_{i,j} \oplus N_{i,j}$$

Prop: $R, S : [m] \nrightarrow [n]$

$$\begin{cases} \underline{\mathrm{mat}}(R \cup S) = \underline{\mathrm{mat}}(R) + \underline{\mathrm{mat}}(S) \\ \underline{\mathrm{mat}}(\emptyset) = 0 \end{cases}$$