BIG

UNIONS and INTERSECTIONS

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathcal{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{c}$	$\neg(\cdot)$
$\bigcup$	$\exists$
$\bigcap$	$\forall$

Example: Big union

- $\mathcal{Z} =^{\text{def}} \{ T \subseteq [5] \mid \text{the sum of the elements of } T \text{ is less than or equal } 2 \}$

$$= \{ \emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\} \}$$

- $\bigcup \mathcal{Z}$ is the union of the sets in $\mathcal{Z}$

$$n \in \bigcup \mathcal{Z} \iff \exists T \in \mathcal{Z}.\, n \in T$$

$$\bigcup \mathcal{Z} = \{0, 1, 2\}$$

Big unions

Definition 90 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$, we let the* big union *(relative to $\mathcal{U}$) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}. \, x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .$$

Examples:

- $\bigcup(\mathcal{P}(U)) = \{x \in U \mid \exists S \in \mathcal{P}(U).\ x \in S\}$

  $= \{x \in U \mid \underline{\text{true}}\} = U$

- $\bigcup \emptyset = \{x \in U \mid \exists S \in \emptyset.\ x \in S\}$

  $= \{x \in U \mid \underline{\text{false}}\} = \emptyset$

ASSOCIATIVITY (idea / intuition)

$\mathcal{F} \subseteq P(P(U))$

$$\{\dots, B, B', \dots\}$$
$$\|$$
$$\mathcal{F} = \{\dots, \mathcal{A}, \mathcal{B}, \dots\}$$
$$\|$$
$$\{\dots, A, A', \dots\}$$

- $\bigcup \mathcal{F} = \dots \cup \mathcal{A} \cup \mathcal{B} \cup \dots$

  $= \{\dots, \dots A, A', \dots, \dots B, B', \dots, \dots\}$

  $\bigcup(\bigcup \mathcal{F}) = (\dots \dots \cup A \cup A' \cup \dots \dots \cup B \cup B' \cup \dots \dots)$

- $\bigcup\{\dots, \bigcup \mathcal{A}, \bigcup \mathcal{B}, \dots\}$

  $= \dots \cup (\dots \cup A \cup A' \cup \dots) \cup (\dots \cup B \cup B' \cup \dots) \cup \dots$

Proposition 91 *For all* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(U)))$,

$$\bigcup\left(\bigcup\mathcal{F}\right) = \bigcup\underbrace{\left\{ \bigcup\mathcal{A} \in \mathcal{P}(U) \;\middle|\; \mathcal{A}\in\mathcal{F} \right\}} \in \mathcal{P}(U) \quad .$$

PROOF:

NB(1): pattern-matching notation for

$$\{ X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}.\ X = \bigcup \mathcal{A} \}$$

NB(2): (Type-checking) as $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(U)))$ we have $\bigcup \mathcal{F} \in \mathcal{P}(\mathcal{P}(U))$ and then $\bigcup(\bigcup\mathcal{F}) \in \mathcal{P}(U)$

$$\bigcup\left(\bigcup \mathcal{F}\right) \overset{\text{RTP}}{=} \bigcup\left\{ \bigcup \mathcal{A} \in \mathcal{P}(U) \mid \mathcal{A} \in \mathcal{F} \right\}$$

$\forall x \in U.$

$x \in \bigcup\left(\bigcup \mathcal{F}\right) \iff x \in \bigcup\left\{ \bigcup \mathcal{A} \in \mathcal{P}(U) \mid \mathcal{A} \in \mathcal{F} \right\}.$

- $x \in \bigcup\left(\bigcup \mathcal{F}\right) \iff \exists S \in \bigcup \mathcal{F}.\ x \in S$

  $\iff \exists S.\ S \in \bigcup \mathcal{F} \wedge x \in S$

  $\iff \exists S.\ \exists \mathcal{A} \in \mathcal{F}.\ S \in \mathcal{A} \wedge x \in S$

- $x \in \bigcup\left\{ \bigcup \mathcal{A} \in \mathcal{P}(U) \mid \mathcal{A} \in \mathcal{F} \right\}$

  $\iff \exists \mathcal{A} \in \mathcal{F}.\ x \in \bigcup \mathcal{A}.$

  $\iff \exists \mathcal{A} \in \mathcal{F}.\ \exists S \in \mathcal{A}.\ x \in S.$

$\boxtimes$

PROOF: For $x \in U$, we show:

$$x \in \bigcup\left(\bigcup \mathcal{F}\right) \iff x \in \bigcup \{ X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}. X = \bigcup \mathcal{A} \}$$

On the one hand,

$$x \in \bigcup\left(\bigcup \mathcal{F}\right) \iff \exists S \in \bigcup \mathcal{F}.\ x \in S$$

$$\iff \exists \mathcal{A} \in \mathcal{F}.\ \exists S \in \mathcal{A}.\ x \in S$$

On the other hand,

$$x \in \bigcup \{ X \in \mathcal{P}(U) \mid \exists \mathcal{A} \in \mathcal{F}.\ X = \bigcup \mathcal{A} \}$$

$$\iff \exists X \in \mathcal{P}(U).\ \exists \mathcal{A} \in \mathcal{F}.\ X = \bigcup \mathcal{A} \wedge x \in X$$

$$\iff \exists \mathcal{A} \in \mathcal{F}.\ x \in \bigcup \mathcal{A}$$

$$\iff \exists \mathcal{A} \in \mathcal{F}.\ \exists S \in \mathcal{A}.\ x \in S.$$

$\boxtimes$

Example: Big intersection

- $\mathcal{S} =^{def} \left\{ S \subseteq [5] \;\middle|\; \begin{array}{l}\text{the sum of the elements}\\ \text{of } S \text{ equals } 6\end{array} \right\}$

$$= \{ \{2,4\}, \{0,2,4\}, \{1,2,3\} \}$$

- $\bigcap \mathcal{S}$ is the intersection of the sets in $\mathcal{S}$

$$n \in \bigcap \mathcal{S} \iff \forall\, S \in \mathcal{S}.\, n \in S$$

$$\bigcap \mathcal{S} = \{2\}$$

Big intersections

Definition 92 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$, we let the* big intersection *(relative to $\mathcal{U}$) be defined as*

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall A \in \mathcal{F}. x \in A \} \quad .$$

Examples:

- $$\bigcap(\mathcal{P}(\mathcal{U})) = \{x \in \mathcal{U} \mid \forall S \in \mathcal{P}(\mathcal{U}).\, x \in S\}$$
  $$= \{x \in \mathcal{U} \mid \underline{\text{false}}\} = \emptyset$$

- $$\bigcap \emptyset = \{x \in \mathcal{U} \mid \forall S \in \emptyset.\, x \in S\}$$
  $$= \{x \in \mathcal{U} \mid \text{true}\} = \mathcal{U}.$$

Theorem 93 *Let*

$$\mathcal{F} = \left\{ S \subseteq \mathbb{R} \,\middle|\, (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \right\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$. *Hence,* $\bigcap \mathcal{F} = \mathbb{N}$.

PROOF:

$\underline{\text{RTP}}$: $\bigcap \mathcal{F} = \mathbb{N} \iff (\bigcap \mathcal{F} \subseteq \mathbb{N} \wedge \mathbb{N} \subseteq \bigcap \mathcal{F})$.

(i) $\underline{\text{RTP}}$: $\bigcap \mathcal{F} \subseteq \mathbb{N}$.

We show $\mathbb{N} \in \mathcal{F}$, which is the case.

(ii) $\underline{\text{RTP}}$: $\mathbb{N} \subseteq \bigcap \mathcal{F}$

$\iff \forall n \in \mathbb{N}.\ n \in \bigcap \mathcal{F} \iff \forall n \in \mathbb{N}.\ \forall S \in \mathcal{F}.\ n \in S.$

We show

$\forall n \in \mathbb{N}.\ P(n)$

where $P(n) =^{def} \forall S \in \mathcal{F}.\ n \in S.$

By induction:

BASE CASE $n=0$: $\forall S \in \mathcal{F}.\ 0 \in S.$ holds by definition of $\mathcal{F}$.

INDUCTIVE STEP. Let $n \in \mathbb{N}$.

(IH) $\forall S \in \mathcal{F}.\ n \in S$

RTP $\forall S \in \mathcal{F}.\ (n+1) \in S.$

Let $S \in \mathcal{F}$. $\implies$ ① $(\forall x \in \mathbb{R}.\ x \in S \Rightarrow (x+1) \in S)$

Then by (I.H.) ② $n \in S$

and so, by ① and ②, $(n+1) \in S$.

$\boxtimes$

Proposition: Let $\mathcal{U}$ be a set and let $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$ be a collection of subsets of $\mathcal{U}$.

(1) For all $S \in \mathcal{P}(\mathcal{U})$,

$S = \bigcup \mathcal{F}$

iff

$$[\forall A \in \mathcal{F} .\, A \subseteq S]$$

and

$$[\forall X \in \mathcal{P}(\mathcal{U}) .\, (\forall A \in \mathcal{F} .\, A \subseteq X) \Rightarrow S \subseteq X]$$

(2) For all $T \in \mathcal{P}(\mathcal{U})$,

$T = \bigcap \mathcal{F}$

iff

$$[\forall A \in \mathcal{F} .\, T \subseteq A]$$

and

$$[\forall Y \in \mathcal{P}(\mathcal{U}) .\, (\forall A \in \mathcal{F} .\, Y \subseteq A) \Rightarrow Y \subseteq T]$$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}.\, x \in X$$

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff \left(\forall X \in \mathcal{F}.\, x \in X\right) \quad .$$

$\{1\} \times A = \{(1,a) \mid a \in A\}$

$\{2\} \times B = \{(2,b) \mid b \in B\}$

$(\{1\} \times A) \cap (\{2\} \times B) = \emptyset$

Disjoint unions

Definition 94 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

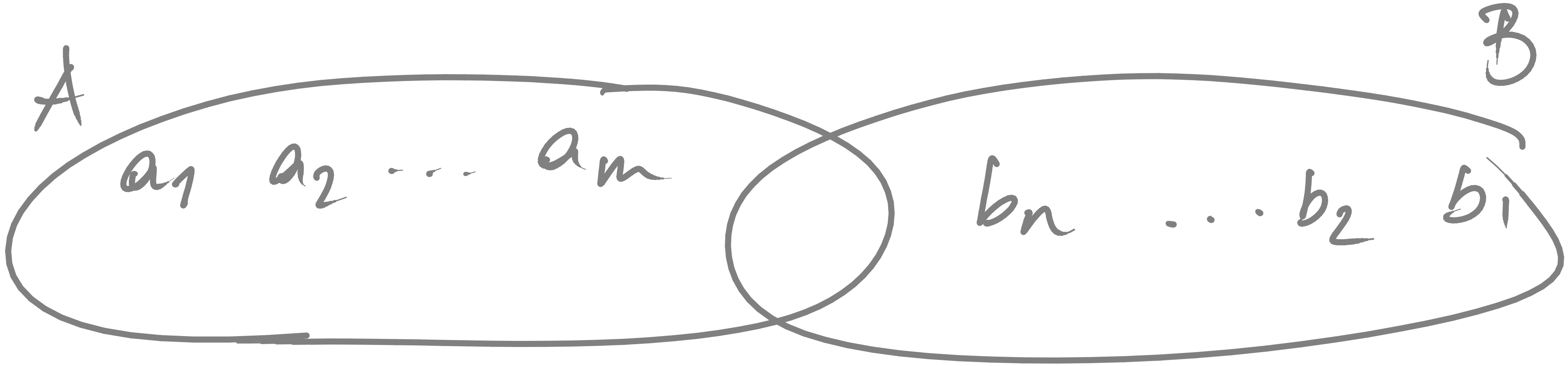
Thus,

$$\forall\, x.\, x \in (A \uplus B) \iff \big(\exists a \in A.\, x = (1,a)\big) \vee \big(\exists b \in B.\, x = (2,b)\big) \,.$$

Proposition 96 *For all finite sets* A *and* B,

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:



Corollary 97 *For all finite sets* A *and* B,

$$\#(A \uplus B) = \#A + \#B \quad .$$

Relations

Definition 99 *A* <u>(binary) relation</u> R *from a set* A *to a set* B

$$R : A \longrightarrow\!\!\!\!\!\!+\!\!\!\!\!\rightarrow B \qquad \textit{or} \qquad R \in \mathrm{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \qquad \textit{or} \qquad R \in \mathcal{P}(A \times B) \quad .$$

Notation 100 *One typically writes* $a\,R\,b$ *for* $(a, b) \in R$*.*

Informal examples:

- Computation.
- Typing.
- Program equivalence.
- Networks.
- Databases.

PROGRAM SEMANTICS

$\underline{Sq} : \mathbb{R}_{\geq 0} \nrightarrow \mathbb{R}$

given by all pairs

(x, y) such that $x = y^2$

In particular, $1 \ \underline{Sq} \ 1$

$1 \ \underline{Sq} \ -1$

TYPING

$$P : \alpha$$

E.g.

$$(\underline{fn}\ x \rightarrow x,\ \underline{bool} \rightarrow bool)$$

$$(\underline{fn}\ x \rightarrow x,\ \underline{nat} \rightarrow nat)$$

are in the Typing relation

$$(\underline{fn}\ x \rightarrow x,\ \underline{bool} \rightarrow \underline{nat})$$

is not.

NETWORKS

N – nodes

C – connections

$$C : N \nrightarrow N$$

DATABASES

A relation R on sets $A_1, A_2, \ldots, A_n$ is defined as a subset

$$R \subseteq A_1 \times A_2 \times \cdots \times A_n$$

E.g. $R \subseteq \text{Movies} \times \text{Directors} \times \text{Years} \times \text{Person}$ consisting of all quadruples (m, d, y, p) such that movie m was directed by director d in year y with person p a cast member.

Examples:

- Empty relation.

  $\emptyset : A \longrightarrow\!\!\!\!\!\!+\,\, B$ $\qquad$ $(a \ \emptyset \ b \iff \mathbf{false})$

- Full relation.

  $(A \times B) : A \longrightarrow\!\!\!\!\!\!+\,\, B$ $\qquad$ $(a \ (A \times B) \ b \iff \mathbf{true})$

- Identity (or equality) relation.

  $\mathrm{id}_A = \{ \, (a, a) \mid a \in A \, \} : A \longrightarrow\!\!\!\!\!\!+\,\, A$ $\qquad$ $(a \ \mathrm{id}_A \ a' \iff a = a')$

- Integer square root.

  $R_2 = \{ \, (m, n) \mid m = n^2 \, \} : \mathbb{N} \longrightarrow\!\!\!\!\!\!+\,\, \mathbb{Z}$ $\qquad$ $(m \ R_2 \ n \iff m = n^2)$