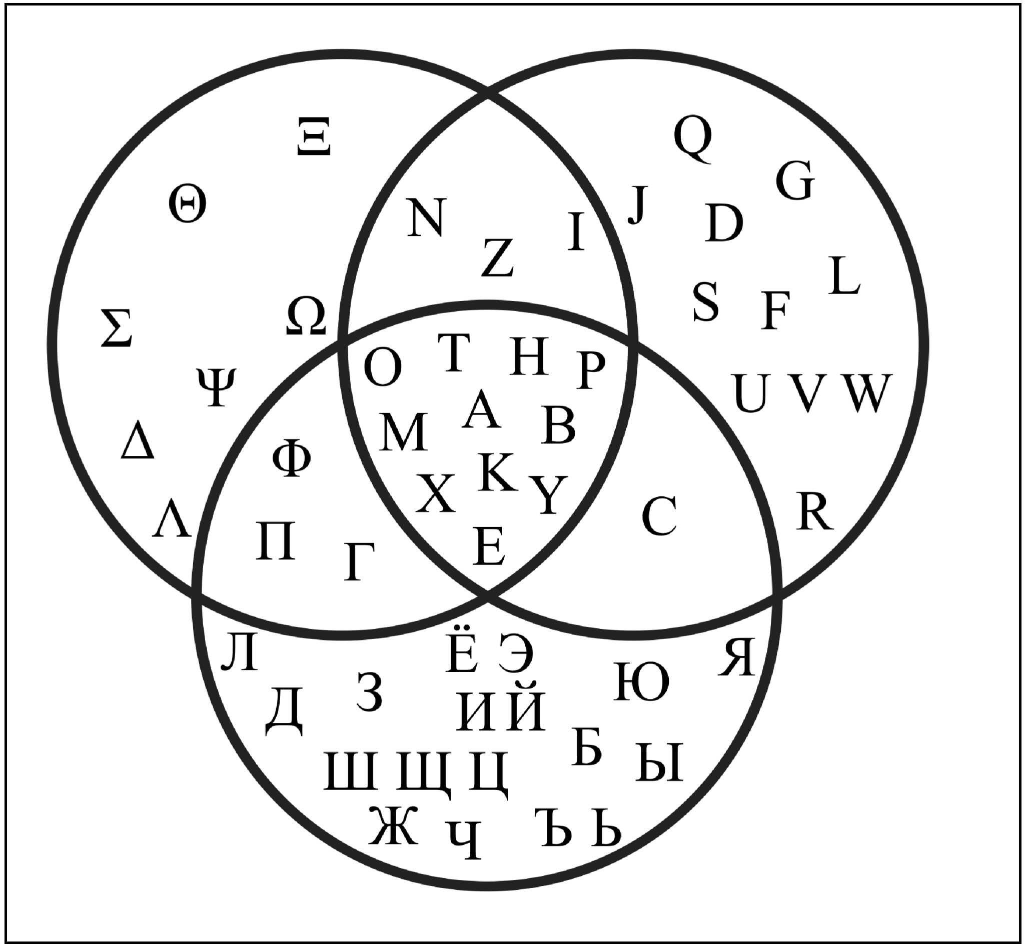
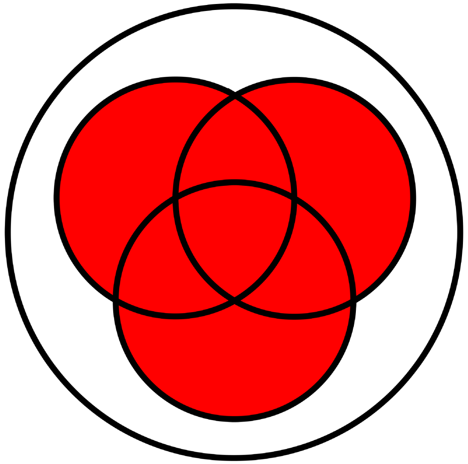
Venn diagrams[a]

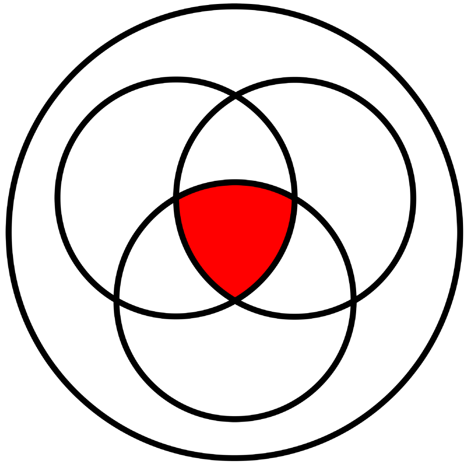


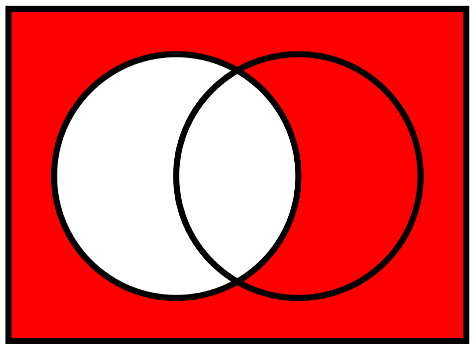
[a]From http://en.wikipedia.org/wiki/Intersection_(set_theory) .

Union



Intersection





Complement

The powerset Boolean algebra

$$(\quad \mathcal{P}(U)\ ,\quad \emptyset\ ,\quad U\ ,\quad \cup\ ,\quad \cap\ ,\quad (\cdot)^c \quad)$$

For all $A, B \in \mathcal{P}(U)$,

$$A \boxed{\cup} B \ = \ \{x \in U \mid x \in A \boxed{\vee} x \in B\} \ \in \mathcal{P}(U)$$

$$A \boxed{\cap} B \ = \ \{x \in U \mid x \in A \boxed{\wedge} x \in B\} \ \in \mathcal{P}(U)$$

$$A^{\boxed{c}} \ = \ \{x \in U \mid \boxed{\neg}(x \in A)\} \ \in \mathcal{P}(U)$$

- The union operation $\cup$ and the intersection operation $\cap$ are associative, commutative, and idempotent.

$$(A \cup B) \cup C = A \cup (B \cup C) \ , \quad A \cup B = B \cup A \ , \quad A \cup A = A$$

$$(A \cap B) \cap C = A \cap (B \cap C) \ , \quad A \cap B = B \cap A \ , \quad A \cap A = A$$

- The *empty set* $\emptyset$ is a neutral element for $\cup$ and the *universal set* $\mathbb{U}$ is a neutral element for $\cap$.

$$\emptyset \cup A \ = \ A \ = \ \mathbb{U} \cap A$$

- The empty set $\emptyset$ is an annihilator for $\cap$ and the universal set $\mathcal{U}$ is an annihilator for $\cup$.

$$\emptyset \cap A = \emptyset$$

$$\mathcal{U} \cup A = \mathcal{U}$$

- With respect to each other, the union operation $\cup$ and the intersection operation $\cap$ are distributive and absorptive.

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad , \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup (A \cap B) = A = A \cap (A \cup B)$$

Prop. $A \cup (A \cap B) = A$

Proof. $\forall x.\ x \in A \cup (A \cap B) \Leftrightarrow x \in A$

Let x be arbitrary.

($\Rightarrow$) Assume $x \in A \cup (A \cap B) \Leftrightarrow (x \in A \vee x \in A \cap B)$

RTP $x \in A$.

Case $x \in A$, we are done.

Case $x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B) \Rightarrow x \in A$

($\Leftarrow$) Assume $x \in A$.

RTP $x \in A \cup (A \cap B)$. which is the case because $x \in A$.

$\square$

- The complement operation $(\cdot)^{\mathrm{c}}$ satisfies complementation laws.

$$A \cup A^{\mathrm{c}} = \mathcal{U} \ , \quad A \cap A^{\mathrm{c}} = \emptyset$$

Proposition 85 *Let* U *be a set and let* $A, B \in \mathcal{P}(U)$.

1. $\forall X \in \mathcal{P}(U).\ A \cup B \subseteq X \iff (A \subseteq X \land B \subseteq X)$.
2. $\forall X \in \mathcal{P}(U).\ X \subseteq A \cap B \iff (X \subseteq A \land X \subseteq B)$.

PROOF: Let $A, B, X \in \mathcal{P}(U)$.

<u>RTP</u>: $A \cup B \subseteq X \iff (A \subseteq X \land B \subseteq X)$

$(\Rightarrow)$ Assume $A \cup B \subseteq X$

<u>RTP</u> $A \subseteq X \land B \subseteq X$

$(\Leftrightarrow)$ $\forall a \in A.\ a \in X \land \forall b \in B.\ b \in X$.

<u>Assume</u> $a \in A$. $\Rightarrow a \in A \cup B \Rightarrow a \in X$

<u>Assume</u> $b \in B \Rightarrow b \in A \cup B \Rightarrow b \in X$.

$(\Leftarrow)$ Assume ① $A \subseteq X$ and ② $B \subseteq X$.

RTP $A \cup B \subseteq X$

Let $x \in A \cup B \iff (x \in A \vee x \in B)$.

Case $x \in A$: Then by ①, $x \in X$.

Case $x \in B$: Then by ②, $x \in X$.

$\boxtimes$

Corollary 86 *Let* U *be a set and let* $A, B, C \in \mathcal{P}(U)$.

1. $$C = A \cup B$$

   iff

   $$[A \subseteq C \wedge B \subseteq C]$$

   $$\wedge$$

   $$[\forall X \in \mathcal{P}(U).\ (A \subseteq X \wedge B \subseteq X) \implies C \subseteq X]$$

   C contains A and B and it is the smallest such.

2. $$C = A \cap B$$

   iff

   $$[C \subseteq A \wedge C \subseteq B]$$

   $$\wedge$$

   $$[\forall X \in \mathcal{P}(U).\ (X \subseteq A \wedge X \subseteq B) \implies X \subseteq C]$$

   C is contained in A and B and it is the biggest such.

Sets and logic

$\mathcal{P}(\mathsf{U})$	$\{\,\mathbf{false}\,,\ \mathbf{true}\,\}$
$\emptyset$	**false**
U	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{\mathrm{c}}$	$\neg(\cdot)$

Unordered & Ordered Pairing

Pairing axiom

> For every a and b, there is a set with a and b as its only elements.

$$\{a, b\} = \{b, a\}$$

defined by

$$\forall x.\, x \in \{a, b\} \iff (x = a \vee x = b)$$

$$\forall x.\, x \in \{b, a\} \iff (x = b \vee x = a)$$

NB The set $\{a, a\}$ is abbreviated as $\{a\}$, and referred to as a *singleton*.

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

Proposition For all a, b, c, x, y,

(1) $\{x, y\} \subseteq \{a\} \Rightarrow (x = a \wedge y = a)$

(2) $\{c, x\} = \{c, y\} \Rightarrow x = y$.

Proof: (1) Assume $\{x, y\} \subseteq \{a\}$

Then since $x \in \{x, y\} \Rightarrow x \in \{a\} \Rightarrow x = a$

Analogously for $y = a$.

(2) Assume $\{c, x\} = \{c, y\}$.

Then
$$\left.\begin{array}{l}(x = c \vee x = y) \\ \wedge\ (y = c \vee y = x)\end{array}\right] \xRightarrow{\text{exercise}} x = y$$

$\boxtimes$

ORDERED PAIRING

Notation:

$$(a,b) \quad \text{or} \quad \langle a,b \rangle$$

Fundamental property:

$$(a,b) = (x,y) \iff (a = x \wedge b = y)$$

Ordered pairing

For every pair a and b, the set

$$\{\{a\},\{a,b\}\}$$

is abbreviated as

$$\langle a, b \rangle$$

and referred to as an *ordered pair*.

Proposition 87 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff (a = x \wedge b = y) .$$

$$\langle a,b\rangle \overset{\text{def}}{=} \{\{a\},\{a,b\}\}$$

PROOF:

($\Leftarrow$) Straightforward.

($\Rightarrow$) Assume

$$\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$$

RTP: $a = x \wedge b = y$.

By assumption,

$$(\{a\}=\{x\} \vee \{a\}=\{x,y\}) \wedge (\{a,b\}=\{x\} \vee \{a,b\}=\{x,y\})$$

$$\big(\{x\} = \{a\} \vee \{x\} = \{a, b\}\big)$$

$$\wedge \big(\{x, y\} = \{a\} \vee \{x, y\} = \{a, b\}\big).$$

Exercise: finish the argument.

$\boxtimes$

Products

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b) \,\}$$

$$= \{ (a,b) \mid a \in A \wedge b \in B \}$$

where

$$\forall a_1, a_2 \in A, b_1, b_2 \in B.$$
$$(a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

PATTERN-MATCHING NOTATION

Example: The subset of ordered pairs from a set A with equal components is formally

$$\{ x \in A \times A \mid \exists a_1 \in A . \exists a_2 \in A . x = (a_1, a_2) \wedge a_1 = a_2 \}$$

but often abbreviated using pattern-matching notation as

$$\{ (a_1, a_2) \in A \times A \mid a_1 = a_2 \} .$$

Notation: For a property $P(a,b)$ with a ranging over a set A and b ranging over a set B,

$$\{(a,b) \in A \times B \mid P(a,b)\}$$

abbreviates

$$\{x \in A \times B \mid \exists a \in A.\, \exists b \in B.\, x = (a,b) \wedge P(a,b)\}.$$

Proposition 89 *For all finite sets* A *and* B,

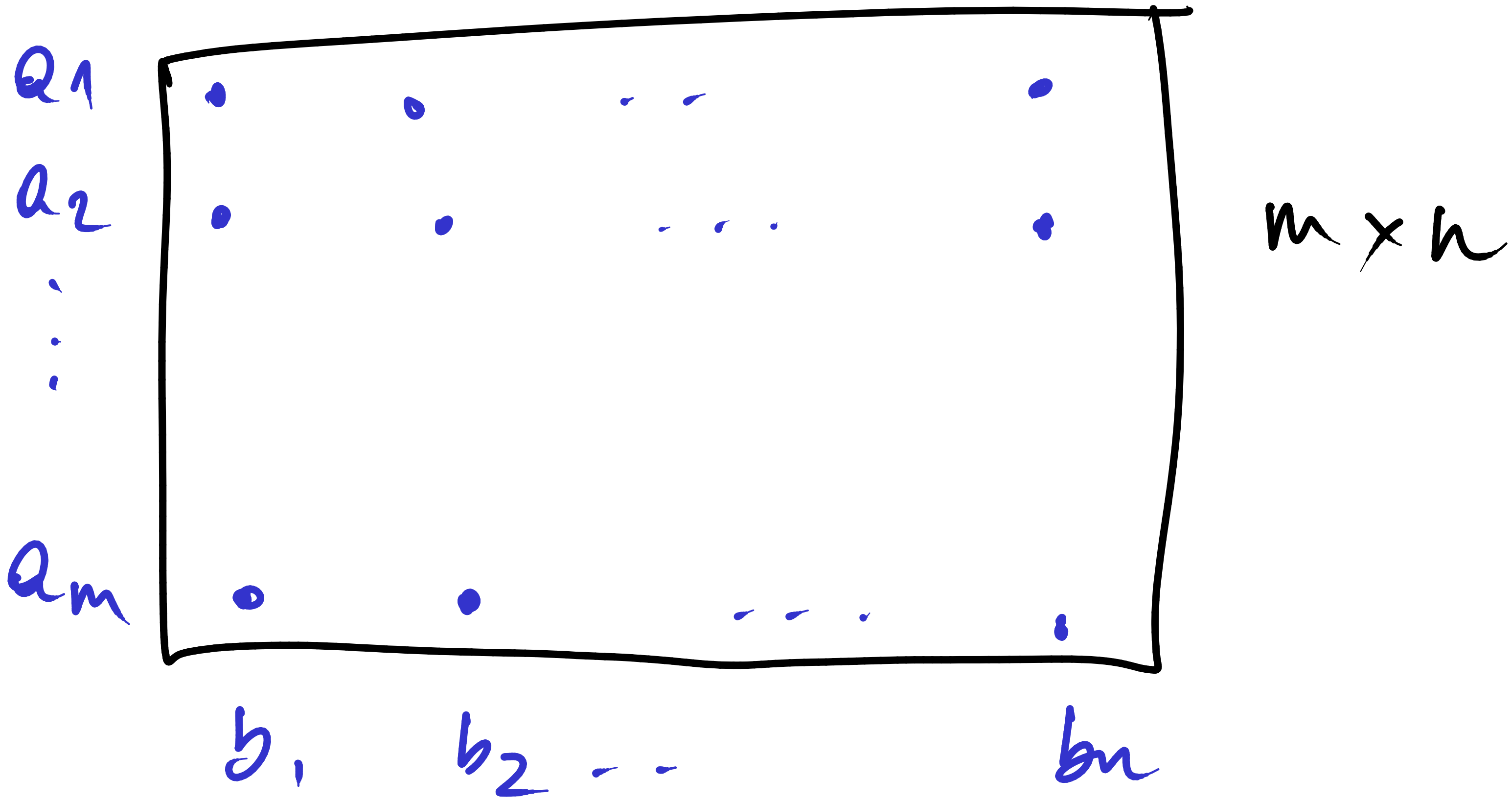
$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA:

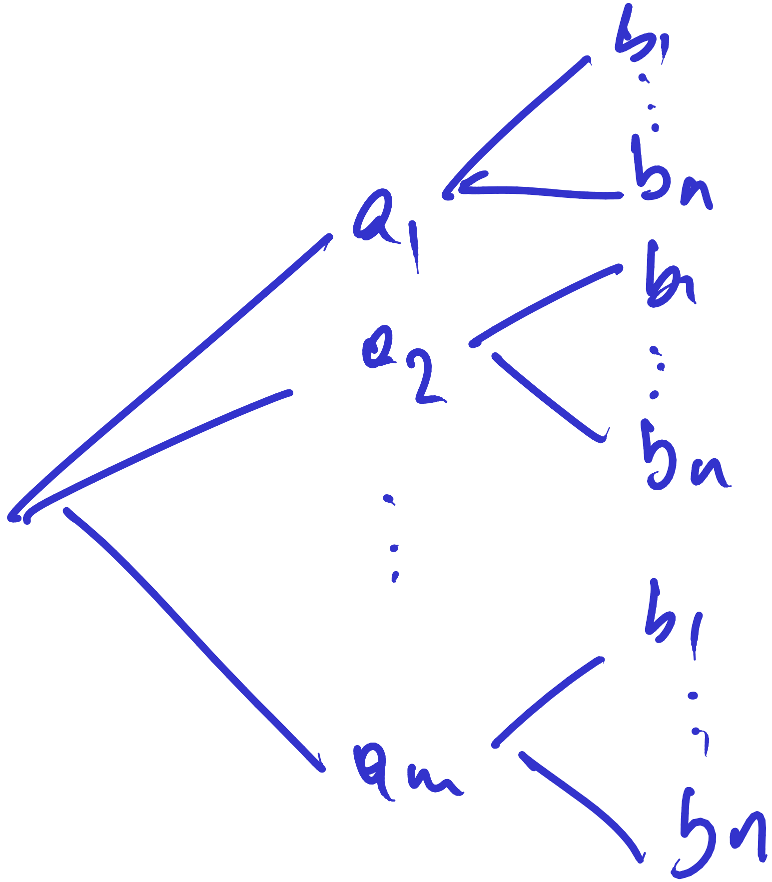
Say $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$

$\#A = m$ $\qquad\qquad$ $\#B = n$

$$\begin{aligned} A \times B = \{ & (a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots, (a_1, b_n), \\ & (a_2, b_1), (a_2, b_2), \ldots, (a_2, b_n), \\ & \ldots \\ & (a_m, b_1), \ldots, (a_m, b_n) \} \end{aligned}$$


a_1
a_2
$\vdots$
a_m
b_1 b_2 ...
b_n
$m \times n$

An element of $A \times B$ is given by an arbitrary element of A, for which I have m choices, and then an arbitrary element of B, for which I have n choices



So, in total $m \times n$ choices.

$\boxtimes$

BIG

UNIONS and INTERSECTIONS

Sets and logic

$\mathcal{P}(\mathbb{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathbb{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^c$	$\neg(\cdot)$
$\bigcup$	$\exists$
$\bigcap$	$\forall$

Example: Big union

- $\mathcal{T} =^{def} \{ T \subseteq [5] \mid$ the sum of the elements of T is less than or equal 2 $\}$

  $= \{ \emptyset, \{0\}, \{1\}, \{0,1\}, \{0,2\} \}$

- $\bigcup \mathcal{T}$ is the union of the sets in $\mathcal{T}$

  $n \in \bigcup \mathcal{T} \iff \exists T \in \mathcal{T}.\, n \in T$

  $\bigcup \mathcal{T} = \{0, 1, 2\}$

Big unions

Definition 90 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$*, we let the* big union *(relative to* $\mathcal{U}$*) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}. \, x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .$$