6. On relations

6.1. Basic exercises

1. Let \( A = \{1, 2, 3, 4\} \), \( B = \{a, b, c, d\} \) and \( C = \{x, y, z\} \).
Let \( R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} : A \rightarrow B \)
and \( S = \{(b, x), (b, y), (c, y), (d, z)\} : B \rightarrow C \).

Draw the internal diagrams of the relations. What is the composition \( S \circ R : A \rightarrow C \)?

2. Prove that relational composition is associative and has the identity relation as the neutral element.

3. For a relation \( R : A \rightarrow B \), let its opposite, or dual relation, \( R^\text{op} : B \rightarrow A \) be defined by:
\[
b R^\text{op} a \iff a R b
\]
For \( R, S : A \rightarrow B \) and \( T : B \rightarrow C \), prove that:
   a) \( R \subseteq S \implies R^\text{op} \subseteq S^\text{op} \)
   b) \( (R \cap S)^\text{op} = R^\text{op} \cap S^\text{op} \)
   c) \( (R \cup S)^\text{op} = R^\text{op} \cup S^\text{op} \)
   d) \( (T \circ S)^\text{op} = S^\text{op} \circ T^\text{op} \)

6.2. Core exercises

1. Let \( R, R' \subseteq A \times B \) and \( S, S' \subseteq B \times C \) be two pairs of relations and assume \( R \subseteq R' \) and \( S \subseteq S' \). Prove that \( S \circ R \subseteq S' \circ R' \).

2. Let \( \mathcal{F} \subseteq \mathcal{P}(A \times B) \) and \( \mathcal{G} \subseteq \mathcal{P}(B \times C) \) be two collections of relations from \( A \) to \( B \) and from \( B \) to \( C \), respectively. Prove that
\[
\left( \bigcup \mathcal{G} \right) \circ \left( \bigcup \mathcal{F} \right) = \bigcup \{ S \circ R | R \in \mathcal{F}, S \in \mathcal{G} \} : A \rightarrow C
\]
Recall that the notation \( \{ S \circ R : A \rightarrow C | R \in \mathcal{F}, S \in \mathcal{G} \} \) is common syntactic sugar for the formal definition \( \{ T \in \mathcal{P}(A \times C) | \exists R \in \mathcal{F}. \exists S \in \mathcal{G}. T = S \circ R \} \). Hence,
\[
T \in \{ S \circ R : A \rightarrow C | R \in \mathcal{F}, S \in \mathcal{G} \} \iff \exists R \in \mathcal{F}. \exists S \in \mathcal{G}. T = S \circ R
\]
What happens in the case of big intersections?

3. Suppose \( R \) is a relation on a set \( A \). Prove that
   a) \( R \) is reflexive iff \( \text{id}_A \subseteq R \)
   b) \( R \) is symmetric iff \( R = R^\text{op} \)
c) $R$ is transitive iff $R \circ R \subseteq R$

d) $R$ is antisymmetric iff $R \cap R^\text{op} \subseteq \text{id}_A$

4. Let $R$ be an arbitrary relation on a set $A$, for example, representing an undirected graph. We are interested in constructing the smallest transitive relation (graph) containing $R$, called the transitive closure of $R$.

a) We define the family of relations which are transitive supersets of $R$:

$$ T_R \triangleq \{ Q : A \to A \mid R \subseteq Q \text{ and } Q \text{ is transitive} \} $$

$R$ is not necessarily going to be an element of this family, as it might not be transitive. However, $R$ is a lower bound for $T_R$, as it is a subset of every element of the family.

Prove that the set $\bigcap T_R$ is the transitive closure for $R$.

b) $\bigcap T_R$ is the intersection of an infinite number of relations so it's difficult to compute the transitive closure this way. A better approach is to start with $R$, and keep adding the missing connections until we get a transitive graph. This can be done by repeatedly composing $R$ with itself: after $n$ compositions, all paths of length $n$ in the graph represented by $R$ will have a transitive connection between their endpoints.

Prove that the (at least once) iterated composition $R^{+} \triangleq R \circ R^{+}$ is the transitive closure for $R$, i.e. it coincides with the greatest lower bound of $T_R$: $R^{+} = \bigcap T_R$. Hint: show that $R^{+}$ is both an element and a lower bound of $T_R$.

7. On partial functions

7.1. Basic exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the sets $\text{PFun}(A_i, A_j)$ for $i, j \in \{2, 3\}$.

   Hint: there may be quite a few, so you can think of ways of characterising all of them without giving an explicit listing.

2. Prove that a relation $R : A \leftrightarrow B$ is a partial function iff $R \circ R^\text{op} \subseteq \text{id}_B$.

3. Prove that the identity relation is a partial function, and that the composition of partial functions is a partial function.

7.2. Core exercises

1. Show that $(\text{PFun}(A, B), \subseteq)$ is a partial order. What is its least element, if it exists?

2. Let $\mathcal{F} \subseteq \text{PFun}(A, B)$ be a non-empty collection of partial functions from $A$ to $B$.

   a) Show that $\bigcap \mathcal{F}$ is a partial function.

   b) Show that $\bigcup \mathcal{F}$ need not be a partial function by defining two partial functions $f, g : A \to B$ such that $f \cup g : A \to B$ is a non-functional relation.

   c) Let $h : A \to B$ be a partial function. Show that if every element of $\mathcal{F}$ is below $h$ then $\bigcup \mathcal{F}$ is a partial function.
Consider a set \( A \).

**Optional advanced exercise**

Say that a relation \( R \).

**Basic exercises**

1. Let \( A_2 = \{1, 2\} \) and \( A_3 = \{a, b, c\} \). List the elements of the sets \( \text{Fun}(A_i, A_j) \) for \( i, j \in \{2, 3\} \).
2. Prove that the identity partial function is a function, and the composition of functions yields a function.
3. Prove or disprove that \( (\text{Fun}(A, B), \subseteq) \) is a partial order.
4. Find endofunctions \( f, g : A \to A \) such that \( f \circ g \neq g \circ f \).

**Core exercises**

1. A relation \( R : A \leftrightarrow B \) is said to be total if \( \forall a \in A. \exists b \in B. a R b \). Prove that this is equivalent to \( \text{id}_A \subseteq R \circ \text{id}_B \). Conclude that a relation \( R : A \leftrightarrow B \) is a function iff \( R \circ \text{id}_B \subseteq \text{id}_A \).
2. Let \( \chi : \mathcal{P}(U) \to (U \Rightarrow [2]) \) be the function mapping subsets \( S \subseteq U \) to their characteristic functions \( \chi_S : U \to [2] \).
   a) Prove that for all \( x \in U \),
      \[
      \chi_{A \cup B}(x) = (\chi_A(x) \lor \chi_B(x)) = \max(\chi_A(x), \chi_B(x))
      \]
      \[
      \chi_{A \cap B}(x) = (\chi_A(x) \land \chi_B(x)) = \min(\chi_A(x), \chi_B(x))
      \]
      \[
      \chi_A(x) = \neg(\chi_A(x)) = (1 - \chi_A(x))
      \]
   b) For what construction \( A?B \) on sets \( A \) and \( B \) does it hold that
      \[
      \chi_{A?B}(x) = (\chi_A(x) \boxplus \chi_B(x)) = (\chi_A(x) + \chi_B(x))
      \]
      for all \( x \in U \), where \( \boxplus \) is the exclusive or operator? Prove your claim.

**Optional advanced exercise**

Consider a set \( A \) together with an element \( a \in A \) and an endofunction \( f : A \to A \).

Say that a relation \( R : \mathbb{N} \leftrightarrow A \) is \((a, f)\)-closed whenever
\[
R(0, a) \quad \text{and} \quad \forall n \in \mathbb{N}, x \in A. R(n, x) \implies R(n + 1, f(x))
\]

Define the relation \( F : \mathbb{N} \leftrightarrow A \) as
\[
F \triangleq \bigcap \{ R : \mathbb{N} \leftrightarrow A \mid R \text{ is \((a, f)\)-closed} \}
\]

a) Prove that \( F \) is \((a, f)\)-closed.

b) Prove that \( F \) is total, that is: \( \forall n \in \mathbb{N}. \exists y \in A. F(n, y) \).

c) Prove that \( F \) is a function \( \mathbb{N} \to A \), that is: \( \forall n \in \mathbb{N}. \exists! y \in A. F(n, y) \).

*Hint:* Proceed by induction. Observe that, in view of the previous item, to show that \( \exists! y \in A. F(k, y) \) it suffices to exhibit an \((a, f)\)-closed relation \( R_k \) such that \( \exists! y \in A. R_k(k, y) \). (Why?) For instance, as the relation \( R_0 = \{(m, y) \in \mathbb{N} \times A \mid m = 0 \implies y = a\} \) is \((a, f)\)-closed one has that \( F(0, y) \implies R_0(0, y) \implies y = a \).
d) Show that if \( h \) is a function \( \mathbb{N} \to A \) with \( h(0) = a \) and \( \forall n \in \mathbb{N}. \ h(n + 1) = f(h(n)) \) then \( h = F \).

Thus, for every set \( A \) together with an element \( a \in A \) and an endofunction \( f : A \to A \) there exists a unique function \( F : \mathbb{N} \to A \), typically said to be \textit{inductively defined}, satisfying the recurrence relation

\[
F(n) = \begin{cases} 
  a & \text{for } n = 0 \\
  f(F(n-1)) & \text{for } n \geq 1 
\end{cases}
\]