Discrete Mathematics

Exercises 4

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4. On induction

4.1. Basic exercises

1. Prove that for all natural numbers $n \geq 3$, if n distinct points on a circle are joined in consecutive order by straight lines, then the interior angles of the resulting polygon add up to $180 \cdot (n-2)$ degrees.

2. Prove that, for any positive integer n, a $2^n \times 2^n$ square grid with any one square removed can be tiled with L-shaped pieces consisting of 3 squares.

4.2. Core exercises

1. Establish the following:

   (a) For all positive integers m and n,

   $$(2^n - 1) \cdot \sum_{i=0}^{m-1} 2^{i \cdot n} = 2^{m \cdot n} - 1$$

   (b) Suppose k is a positive integer that is not prime. Then $2^k - 1$ is not prime.

2. Prove that

   $$\forall n \in \mathbb{N}.\ \forall x \in \mathbb{R}.\ x \geq -1 \implies (1+x)^n \geq 1 + n \cdot x$$

3. Recall that the Fibonacci numbers F_n for $n \in \mathbb{N}$ are defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$ for $n \in \mathbb{N}$.

   a) Prove Cassini's Identity: For all $n \in \mathbb{N}$,

   $$F_n \cdot F_{n+2} = {F_{n+1}}^2 + (-1)^{n+1}$$

   b) Prove that for all natural numbers k and n,

   $$F_{n+k+1} = F_{n+1} \cdot F_{k+1} + F_n \cdot F_k$$

   c) Deduce that $F_n \mid F_{l \cdot n}$ for all natural numbers n and l.

   d) Prove that $\mathtt{gcd}(F_{n+2}, F_{n+1})$ terminates with output 1 in n steps for all positive integers n.

   e) Deduce also that:

      (i) for all positive integers $n < m$, $\gcd(F_m, F_n) = \gcd(F_{m-n}, F_n)$,

      and hence that:

      (ii) for all positive integers m and n, $\gcd(F_m, F_n) = F_{\gcd(m,n)}$.

f) Show that for all positive integers m and n, $(F_m \cdot F_n) \mid F_{m \cdot n}$ if $\gcd(m, n) = 1$.

g) Conjecture and prove theorems concerning the following sums for any natural number n:

(i) $\sum_{i=0}^{n} F_{2\cdot i}$

(ii) $\sum_{i=0}^{n} F_{2\cdot i+1}$

(iii) $\sum_{i=0}^{n} F_i$

4.3. Optional exercises

1. Recall the `gcd0` function from §3.3.3. Use the Principle of Mathematical Induction from basis 2 to formally establish the following correctness property of the algorithm:

For all natural numbers $l \geq 2$, we have that for all positive integers m, n, if $m + n \leq l$ then `gcd0`(m, n) terminates.

2. The set of *univariate polynomials* (over the rationals) on a variable x is defined as that of arithmetic expressions equal to those of the form $\sum_{i=0}^{n} a_i \cdot x^i$, for some $n \in \mathbb{N}$ and some *coefficients* $a_0, a_1, \ldots, a_n \in \mathbb{Q}$.

(a) Show that if $p(x)$ and $q(x)$ are polynomials then so are $p(x) + q(x)$ and $p(x) \cdot q(x)$.

(b) Deduce as a corollary that, for all $a, b \in \mathbb{Q}$, the linear combination $a \cdot p(x) + b \cdot q(x)$ of two polynomials $p(x)$ and $q(x)$ is a polynomial.

(c) Show that there exists a polynomial $p_2(x)$ such that $p_2(n) = \sum_{i=0}^{n} i^2 = 0^2 + 1^2 + \cdots + n^2$ for every $n \in \mathbb{N}$.[1]

Hint: Note that for every $n \in \mathbb{N}$,

$$(n+1)^3 = \sum_{i=0}^{n}(i+1)^3 - \sum_{i=0}^{n} i^3$$

(d) Show that, for every $k \in \mathbb{N}$, there exists a polynomial $p_k(x)$ such that, for all $n \in \mathbb{N}$, $p_k(n) = \sum_{i=0}^{n} i^k = 0^k + 1^k + \cdots + n^k$.

Hint: Generalise the hint above, and the similar identity

$$(n+1)^2 = \sum_{i=0}^{n}(i+1)^2 - \sum_{i=0}^{n} i^2$$

[1] Chapter 2.5 of *Concrete Mathematics* by R.L. Graham, D.E. Knuth and O. Patashnik looks at this in great detail.