

Discrete Mathematics

Exercises 7

Marcelo Fiore Ohad Kammar Dima Szamozvancev

9. On bijections

9.1. Basic exercises

- Define a function that has (i) none, (ii) exactly one, and (iii) more than one retraction.
 - Define a function that has (i) none, (ii) exactly one, and (iii) more than one section.
- Let n be an integer.
 - How many sections are there for the absolute-value map $x \mapsto |x|: [-n..n] \rightarrow [0..n]$?
 - How many retractions are there for the exponential map $x \mapsto 2^x: [0..n] \rightarrow [0..2^n]$?
- Give an example of two sets A and B and a function $f: A \rightarrow B$ such that f has a retraction but no section. Explain how you know that f has these properties.
- Prove that the identity function is a bijection and that the composition of bijections is a bijection.
- For $f: A \rightarrow B$, prove that if there are $g, h: B \rightarrow A$ such that $g \circ f = \text{id}_A$ and $f \circ h = \text{id}_B$ then $g = h$. Conclude as a corollary that, whenever it exists, the inverse of a function is unique.

9.2. Core exercises

- We say that two functions $s: A \rightarrow B$ and $r: B \rightarrow A$ are a *section-retraction pair* whenever $r \circ s = \text{id}_A$; and that a function $e: B \rightarrow B$ is an *idempotent* whenever $e \circ e = e$. This question demonstrates that section-retraction pairs and idempotents are closely connected: any section-retraction pair gives rise to an idempotent function, and any idempotent function can be split into a section-retraction pair.
 - Let $f: C \rightarrow D$ and $g: D \rightarrow C$ be functions such that $f \circ g \circ f = f$.
 - Can you conclude that $f \circ g$ is idempotent? What about $g \circ f$? Justify your answers.
 - Define a map g' using f and g that satisfies both

$$f \circ g' \circ f = f \quad \text{and} \quad g' \circ f \circ g' = g'$$

- Show that if $s: A \rightarrow B$ and $r: B \rightarrow A$ are a section-retraction pair then the composite $s \circ r: B \rightarrow B$ is idempotent.
- Show that for every idempotent $e: B \rightarrow B$ there exists a set A (called a *retract* of B) and a section-retraction pair $s: A \rightarrow B$ and $r: B \rightarrow A$ such that $s \circ r = e$.

10. On equivalence relations

10.1. Basic exercises

1. Prove that the isomorphism relation \cong between sets is an equivalence relation.
2. Prove that the identity relation id_A on a set A is an equivalence relation, and that $A/\text{id}_A \cong A$.
3. Show that, for a positive integer m , the relation \equiv_m on \mathbb{Z} given by

$$x \equiv_m y \iff x \equiv y \pmod{m}$$

is an equivalence relation. What are the equivalence classes of this relation?

4. Show that the relation \equiv on $\mathbb{Z} \times \mathbb{Z}^+$ given by

$$(a, b) \equiv (x, y) \iff a \cdot y = x \cdot b$$

is an equivalence relation. What are the equivalence classes of this relation?

10.2. Core exercises

1. Let E_1 and E_2 be two equivalence relations on a set A . Either prove or disprove the following statements.
 - a) $E_1 \cup E_2$ is an equivalence relation on A .
 - b) $E_1 \cap E_2$ is an equivalence relation on A .
2. For an equivalence relation E on a set A , show that $[a_1]_E = [a_2]_E$ iff $a_1 E a_2$, where

$$[a]_E = \{x \in A \mid x E a\}.$$

3. For a function $f : A \rightarrow B$ define a relation \equiv_f on A by the rule: for all $a, a' \in A$,

$$a \equiv_f a' \iff f(a) = f(a')$$

- a) Show that for every function $f : A \rightarrow B$, the relation \equiv_f is an equivalence relation on A .
- b) Prove that every equivalence relation E in a set A is equal to \equiv_q , where $q : A \rightarrow A/E$ is the quotient function $q(a) = [a]_E$.
- c) Prove that for every surjection $f : A \twoheadrightarrow B$,

$$B \cong (A / \equiv_f)$$